Universität Hannover
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On Multi-Instanton and Multi-Monopole Configurations in Yang-Mills Theory

Diplomarbeit

von

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Contents

1 Introduction

Instanton and monopole solutions are of wide interest even though they have never been observed experimentally up to now. Instantons and monopoles are quite similar. They are finite-action or finite-energy solutions in $4+0$ or $3+1$ dimensional gauge theory. Their construction is related by dimensional reduction. Besides a lot of nonperturbative effects in pure Yang-Mills and Yang-Mills-Higgs theory, respectively, they arise as solutions to the effective field equations in string theory. In connection with string theory, it is sometimes necessary to study them on noncommutative spaces. Moreover, such kind of spaces are very interesting in their own right and the idea of noncommutative coordinates is rather old [47].

Instanton and monopole solutions have been known for quite a long time. The story started in 1931, when Dirac found his famous magnetic monopole solutions [15], and in 1975, when instanton solutions were discovered by Belavin, Polyakov, Schwarz and Tyupkin [6]. Since then a lot of work was done in constructing explicit solutions and investigating applications in various fields. The first instanton solutions in noncommutative coordinates were given by Nekrasov and Schwarz [35] in 1998.

This diploma thesis is organized as follows. In section 2 we will provide some mathematical background for the discussions in the following chapters. Chapters 3 and 4 are devoted to instantons in commutative and noncommutative coordinates, respectively. In both chapters we will use a method developed by Atiyah, Hitchin, Drinfeld and Manin [2] in order to construct instanton solutions, which was deformed to noncommutative coordinates by Nekrasov and Schwarz [35]. Before we will introduce and apply this construction in chapter 3, we shall give a short review on instantons and present the famous 't Hooft solution. Then we will construct instanton solutions that have no singularities at finite points. Furthermore, we will explain how the singular 't Hooft solutions are related to solutions regular at finite points by some singular gauge transformations. We shall start chapter 4 with an introduction to the noncommutative space $\mathbb{R}^4_\theta$. Afterwards, we will outline a problem which occurs in the construction of 't Hooft instantons in noncommutative coordinates and show how to handle it. In the end of this chapter, we will try to generalize the construction of nonsingular instanton solutions to noncommutative coordinates - at least for the case of two instantons.

Chapters 5 and 6 are devoted to monopoles in commutative and noncommutative coordinates, respectively. The method of Atiyah, Hitchin, Drinfeld and Manin was adapted by Nahm for the construction of monopole solutions. But instead of this we will use the so-called splitting approach in order to obtain monopole solutions. On the other hand, the splitting approach can also be used for the construction of instanton solutions (see e.g. [24, 31]). We shall start chapter 5 with a general introduction to monopoles. Then we will describe their construction via the splitting approach. We will proceed with investigating the construction of Dirac monopoles and Wu-Yang [57] monopoles. After some introduction to the noncommutative $\mathbb{R}^3_\theta$ in chapter 6, we will consider the generalization of the construction of Wu-Yang and Dirac monopoles to noncommutative coordinates.

2 Fiber Bundles and Gauge Theory

In this chapter we want to give a short review on fiber bundles, including local sections and transition functions, and we will study their geometry using connection and curvature. This will be important for the mathematical understanding of gauge theories. Furthermore, we will introduce homotopy groups and homotopy classes, which we will also need in our discussion of instantons and monopoles. This section is mainly based on [14, 33, 17, 56]. We are not able to provide any proofs here, but they can be found in these references. For more details on differential geometry (forms, tensors, Lie groups and Lie algebras etc.) and algebraic topology see e.g. [33, 17, 28, 8, 25, 23].

Principal bundles. Let M be a manifold, which we will call the base space, and let G be a Lie group. A differentiable principal bundle P over M with structure group G, satisfies the following conditions:

- G acts freely on P on the right $(p, g) \in P \times G \mapsto p\ g \in P$, i.e. if there is any $p \in P$ such that $p\ g = p$, then g is equal to the identity element e of G.
- M is equal to the quotient space $M = P/G$, where the equivalence relation is induced by the action of G on P. The surjection $\pi : P \to M$ is differentiable and is called the projection.
- P is locally trivial, i.e. every point $x \in M$ has a neighborhood $U \subset M$ and there exists a diffeomorphism $\phi : \pi^{-1}(U) \to U \times G$ such that $\phi(u) = (\pi(u), \varphi(u))$ for $u \in \pi^{-1}(U)$, where $\varphi : \pi^{-1}(U) \to G$ satisfies $\varphi(u\ g) = \varphi(u)\ g$ for all $u \in \pi^{-1}(U)$ and $g \in G$.

For $x \in M$, $\pi^{-1}(x)$ is called the fiber over x. If $u \in \pi^{-1}(x)$, then

$$\pi^{-1}(x) = \{ug \mid g \in G\}. \tag{2.1}$$

Every fiber is a submanifold of P and diffeomorphic to G.

A principal bundle P is called trivial if P is a product space $M \times G$. Then for $(x, g) \in M \times G$ the right action is simply given by $(x, g)a = (x, ga)$.

Local sections. A local section of a principal bundle P over an open set $U \subset M$ is a map $\sigma : U \to P$ such that $\pi(\sigma(x)) = x$ for all $x \in U$. One can always construct a local section on $U \subset M$ which is diffeomorphic to $U \times \{e\}$ by

$$\sigma(x) = u\ (\varphi(u))^{-1}. \tag{2.2}$$

This construction is independent of the choice of the point u in the fiber $\pi^{-1}(x)$ since any point $v \in \pi^{-1}(x)$ can be written as $v = ug$ for $g \in G$ and we have

$$v\ (\varphi(v))^{-1} = u\ g\ (\varphi(u)\ g)^{-1} = u\ (\varphi(u))^{-1}.$$

One calls (2.2) the preferred local section over U, which we will use in the following discussions.

A principal bundle P has a global section, i.e. $U = M$, if and only if P is trivial.

Transition functions. Let $\{U_\alpha\}$ be an open covering of the base space M and let P be a principal bundle over M with a structure group G. Suppose $x \in U_\alpha \cap U_\beta \neq \emptyset$. We will use the preferred local section (2.2) over U_α to construct the preferred local section over U_β

$$\sigma_\beta(x) = \sigma_\alpha(x)\ \varphi_\alpha(u)\ (\varphi_\beta(u))^{-1} = u\ (\varphi_\beta(u))^{-1}.$$

We define

$$\phi_{\alpha\beta}(x) = \varphi_\alpha(u)\ (\varphi_\beta(u))^{-1} \tag{2.3}$$

on $U_\alpha \cap U_\beta$. These maps $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ are called transition functions. They satisfy the so-called cocycle condition

$$\phi_{\alpha\beta}(x) = \phi_{\alpha\gamma}(x)\ \phi_{\gamma\beta}(x) \text{ for all } x \in U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset. \tag{2.4}$$

Principal coordinate bundle. Let P be a principal bundle over M with a structure group G. Consider the transition functions (2.3) constructed from the preferred local sections (2.2) over the covering $\{U_\alpha\}$ of M. One says that $(M, G, \{U_\alpha\}, \phi_{\alpha\beta})$ describes a principal coordinate bundle.

A different set of local sections over $\{U_\alpha\}$ can be obtained from

$$\sigma'_\alpha(x) = \sigma_\alpha(x)\ g(x),$$

where $g(x) \in G$. On the intersection of U_α and U_β the new local sections are related by

$$\sigma'_\beta(x) = \sigma'_\alpha(x)\ g^{-1}(x)\ \varphi_\alpha(u)\ (\varphi_\beta(u))^{-1}\ g(x) \qquad \text{for } x \in U_\alpha \cap U_\beta \neq \emptyset.$$

Therefore, we have the following relation for the new transition functions:

$$\phi'_{\alpha\beta}(x) = g^{-1}(x)\ \phi_{\alpha\beta}(x)\ g(x). \tag{2.5}$$

One says that the two principal coordinate bundles, $(M, G, \{U_\alpha\}, \phi_{\alpha\beta})$ and $(M, G, \{U_\alpha\}, \phi'_{\alpha\beta})$, are equivalent. A principal bundle is therefore the equivalence class of its principal coordinate bundles. Furthermore, a theorem exists that any set of functions $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to G$, which are defined for a covering $\{U_\alpha\}$ of the manifold M and satisfy the cocycle condition (2.4), completely determines the principal bundle.

Associated fiber bundles. Let again P be a principal bundle over M with a structure group G. We will construct an associated fiber bundle as follows. Let F be a manifold on which G acts on the left. Define the action of $g \in G$ on $P \times F$ by $(p, f) \to (p\ g, g^{-1}f)$. The associated fiber bundle E, with the standard fiber F, is an equivalence class $P \times F/G$ in which two points (u, f) and $(p\ g, g^{-1}f)$ are identified. Then the projection $\pi_E : E \to M$ of the bundle E is given by

$$\pi_E((u, f)) = \pi(u).$$

Connection. Let P be a principal bundle over M with a structure group G and let $\mathfrak{g}$ be the Lie algebra of G. Through any point $p \in P$, one can draw the curve

$$p_t = p\ \exp(tX)$$

for $X \in \mathfrak{g}$. This curve lies in the fiber over $\pi(p)$. Now one can define a vector field $X^\sharp$ on P associated to any $X \in \mathfrak{g}$ by

$$X_p^\sharp\, f = \frac{\mathrm{d}}{\mathrm{d}t} f(p_t)|_{t=0},$$

where $f : P \to \mathbb{R}$ is an arbitrary smooth function. The vector $X_p^\sharp$ is tangent to P at p and hence it is tangent to the fiber over $\pi(p)$. One calls the space of all vectors tangent to the fiber through $\pi(p)$ the vertical subspace and denote it by V_pP. Clearly, V_pP is a subspace of the tangent space T_pP. Note that the map $\sharp : \mathfrak{g} \to V_pP$ given by $X \mapsto X^\sharp$ is a vector space isomorphism.

A connection on P is a unique separation of T_pP into V_pP and H_pP, the horizontal subspace, such that

- $T_pP = V_pP \oplus H_pP$,
- a smooth vector field X on P is separated into smooth vector fields $X^H \in H_pP$ and $X^V \in V_pP$ as $X = X^H + X^V$ and
- H_pP and $H_{pg}P$ on the same fiber are related by a linear map R_{g*} induced by the right action R_g of G on P by $H_{pg}P = R_{g*}\, H_pP$

for arbitrary $p \in P$ and $g \in G$. This separation can be realized with the help of a connection one-form. Let ω be a Lie-algebra valued one-form such that

- $\omega(X^\sharp) = X \qquad X \in \mathfrak{g}$
- $(R_g^*\omega)(X) = \mathrm{Ad}_{g^{-1}}\, \omega(X) \qquad X \in \mathfrak{g}$ and $g \in G$.

Let now $\{U_\alpha\}$ be a covering of M, σ_α the preferred local sections on U_α and $\phi_{\alpha\beta}$ the transition functions. On each U_α one can define a Lie-algebra valued one-form by

$$A_\alpha = \sigma_\alpha^*\, \omega.$$

The local forms A_α satisfy the compatibility condition

$$A_\beta = \mathrm{Ad}_{\phi_{\alpha\beta}^{-1}}\, A_\alpha + \phi_{\alpha\beta}^{-1}\, \mathrm{d}\phi_{\alpha\beta} \tag{2.6}$$

on $U_\alpha \cap U_\beta \neq \emptyset$ (d denotes the exterior derivative on M) and are called gauge potentials.

On the other hand, for a given set of local sections satisfying the compatibility condition (2.6), there exists a unique connection one-form ω on P. The local form A_α is said to be the gauge potential on U_α.

Let now x_α^μ be local coordinates on U_α. Then one can express A_α in terms of its components $A_{\alpha\mu}(x) \in \mathfrak{g}$, which are Lie-algebra valued functions

$$A_\alpha = A_{\alpha\mu}(x)\ \mathrm{d}x_\alpha^\mu. \tag{2.7}$$

This is obviously section dependent. On another section $\sigma'_\alpha(x) = \sigma_\alpha(x)\, g(x)$ with $g : U_\alpha \to G$ one has $A'_\alpha = A'_{\alpha\mu}\ \mathrm{d}x_\alpha^\mu$ with

$$A'_{\alpha\mu}(x) = (g(x))^{-1}\ A_{\alpha\mu}(x)\ g(x) + (g(x))^{-1}\ \partial_\mu g(x) \tag{2.8}$$

Covariant derivative. Let P be a principal bundle over M with a structure group G and let ω be a connection on P. The horizontal lift of a vector field X on M is the unique horizontal field $\tilde{X}$ on P which projects onto X

$$\pi_*(\tilde{X}_p) = X_{\pi(p)} \qquad \text{for all } p \in P.$$

Let $A_{\alpha\mu}(x)$ be the components of $A_\alpha = \sigma_\alpha^*\, \omega$ on U_α , where $\{U_\alpha\}$ is the covering of M and σ_α a local section on U_α. The lift of a vector field ∂_μ on U_α is then

$$\tilde{\partial}_\mu|_{\sigma_\alpha(x)} = \sigma_{\alpha *}\partial_\mu - A^\sharp_{\alpha\mu}$$

since $\omega(\sigma_{\alpha *}\partial_\mu - A^\sharp_{\alpha\mu}) = 0$. One can identify $\sigma_{\alpha *}\partial_\mu$ with ∂_μ, $A^\sharp_{\alpha\mu}$ with $A_{\alpha\mu}$ and the horizontal lift $\tilde{\partial}_\mu$ with the covariant derivative on U_α

$$D_\mu = \partial_\mu - A_{\alpha\mu}. \tag{2.9}$$

This is again section dependent. On some other point $\sigma_\alpha(x)\; g(x)$ on the same fiber (again $g : U_\alpha \to G$) one has $D'_\mu = \partial_\mu - A'_\mu$ with $A'_{\alpha\mu}$ exactly as in (2.8).

Curvature. Let P be a principal bundle over M with a structure group G and let ω be a connection on P. The curvature form F is defined to be

$$F = D\omega = \mathrm{d}\omega + \omega \wedge \omega = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega]. \tag{2.10}$$

Here D denotes the exterior covariant derivative and is defined as follows: if ψ is an r-form on a principal fiber bundle P endowed with a connection, then

$$D\psi(X_1, \dots, X_{r+1}) = \mathrm{d}\psi(X_1^H, \dots, X_{r+1}^H).$$

One has the Bianchi identity

$$D\Omega = 0 \tag{2.11}$$

for Ω any Lie-algebra valued two-form.

Let $\{U_\alpha\}$ be a covering of M, σ_α the preferred local sections (2.2) on U_α and $\phi_{\alpha\beta}$ the transition functions. On each U_α one can define a Lie-algebra valued two-form by

$$F_\alpha = \sigma_\alpha^*\; F$$

restricted on U_α. On the intersection of two charts, $U_\alpha \cap U_\beta \neq \emptyset$, the two-forms are related by

$$F_\beta = \mathrm{Ad}_{\phi^{-1}_{\alpha\beta}} F_\alpha. \tag{2.12}$$

We consider again local coordinates x^μ on U_α. One can express F_α in terms of its components as

$$F_\alpha = \frac{1}{2}\; F_{\alpha\mu\nu}\; \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu. \tag{2.13}$$

Then one finds

$$F_{\alpha\mu\nu} = \partial_\mu A_{\alpha\nu} - \partial_\nu A_{\alpha\mu} + [A_{\alpha\mu}, A_{\alpha\nu}] \tag{2.14}$$

on U_α. These components $F_{\alpha\mu\nu}$ are called field strength.

Gauge transformations. Let P be a principal bundle over M with a structure group G and let ω be a connection on P. A gauge transformation on P is a mapping $f : P \to P$ such that

- for all $p \in P$ there exists $g(p) \in G$ such that $f(p) = p\, g(p)$ and
- $g(pa) = a^{-1}\, g(p)\, a$ for all $p \in P$ and $a \in G$.

The action of the gauge transformation on a connection form is the one induced by the automorphism on P. This is a global definition of the gauge transformation and therefore coordinate independent.

Suppose the bundle P has a global section, i.e. P is trivial, then gauge transformation can be viewed as a transformation from one section σ to another section $\sigma' = \sigma\, \gamma$ with $\gamma : M \to G$. The components of the gauge potential $A = A_\mu\, \mathrm{d}x^\mu$ transform like

$$A'_\mu = \mathrm{Ad}_\gamma\, A_\mu + \gamma \partial_\mu \gamma^{-1}. \tag{2.15}$$

Homotopy theory. Let X, Y be two topological spaces and $f : X \to Y$, $g : X \to Y$ two continuous maps. One says that f is homotopic to g if there is a continuous map $F : X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all $x \in X$. The map F is called a homotopy. This defines an equivalence relation on the space of all maps from X to Y and the equivalence classes $[f]$ are called homotopy classes.

Let us now consider $X = [0,1]$. The map $f : [0,1] \to Y$ describes a path in Y and is called a loop if $f(0) = f(1) = y_0$. The set of all homotopy classes $\pi_1(Y, y_0) = \{[f] \mid f : [0,1] \to Y\ ,\ f(0) = f(1) = y_0\}$ form a group under the multiplication $[f] \cdot [g] = [f \cdot g]$, where

$$(f \cdot g)(t) = \begin{cases} f(2t) & \text{for } 0 \le t \le \frac{1}{2}, \\ g(2t-1) & \text{for } \frac{1}{2} \le t \le 1 \end{cases} . \tag{2.16}$$

Let Y be path connected. Then the two groups with different base points $\pi_1(Y, y_0)$ and $\pi_1(Y, y_1)$ are isomorphic. Therefore $\pi(Y)$ is independent of the base point and is called the fundamental group of Y. Let us consider S^1, the one-sphere, as an example. One obtains $\pi_1(S^1) \cong \mathbb{Z}$, the winding number. This number shows how often $[0,1]$ is wrapped around the one-sphere.

One can extend this concept to higher homotopy groups. Let $I^n = [0,1] \times \ldots \times [0,1]$ be the unit n cube. Now a loop is defined as f takes the same value y_0 on the boundary ∂I^n. Then the set of all homotopy classes $\pi_n(Y, y_0) = \{[f] \mid f : I^n \to Y,\ f \text{ is a loop}\}$ forms again a group under the multiplication $[f] \cdot [g] = [f \cdot g]$, where

$$(f \cdot g)(t_1, \ldots, t_n) = \begin{cases} f(2t_1, t_2, \ldots, t_n) & \text{for } 0 \le t_1 \le \frac{1}{2}, \\ g(2t_1 - 1, t_2, \ldots, t_n) & \text{for } \frac{1}{2} \le t_1 \le 1 \end{cases} . \tag{2.17}$$

If Y is again path connected the nth homotopy group $\pi_n(y)$ is independent of the base point. And, as an example, one can see that $\pi_n(S^n) \cong \mathbb{Z}$, the winding number. This number shows how often I^n is wrapped around S^n. The following table gives some examples of other homotopy groups.

Y	π_1	π_2	π_3
$SO(3)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$
$SO(4)$	$\mathbb{Z}_2$	0	$\mathbb{Z} + \mathbb{Z}$
$U(1)$	$\mathbb{Z}$	0	0
$SU(2)$	0	0	$\mathbb{Z}$
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$
S^3	0	0	$\mathbb{Z}$

(2.18)

Classification of principal bundles. Let P be a principal bundle over M with a structure group G. Consider a map $f : N \to M$, where N is some manifold. One can construct the pull-back bundle $f^*(P)$ over N, by gluing over any point $y \in N$ a copy of the fiber $\pi^{-1}(x)$ over $x \in M$. There exists the following theorem: for each principal bundle over a paracompact space M with a structure group G there exists a universal bundle PG over MG with the structure group G such that

- for each principal bundle P over M with a structure group G there exists a map $f : M \to MG$ such that P and the pull-back bundle $f^*(PG)$ are isomorphic and
- if $f_1, f_2 : M \to MG$ are two maps such that the pull-back bundles $f_1^*(PG)$ and $f_2^*(PG)$ are isomorphic, then f_1 and f_2 are homotopic.

Therefore we have translated the classification of principal bundles with a structure group G into finding homotopy classes of maps $f : M \to MG$. For G a compact, simple and simply connected Lie group one has the short exact sequence

$$0 \to \pi_4(MG) \to \pi_3(G) \to 0. \tag{2.19}$$

An exact sequence is a sequence of homomorphisms

$$\ldots \to X_{n+1} \xrightarrow{f_{n+1}} X_n \xrightarrow{f_n} X_{n-1} \to \ldots, \tag{2.20}$$

where the image of f_{n+1} is equal to the kernel of f_n for all n. A short exact sequence is an exact sequence of the form

$$0 \to X \to Y \to 0 \tag{2.21}$$

Note that in this case the homomorphism $X \to Y$ must be an isomorphism.

As an example of (2.19) one obtains that the set of equivalence classes of principal bundles over S^4 with a compact, simple and simply connected structure group G is isomorphic to $\pi_4(S^4G) \cong \pi_3(G) \cong \mathbb{Z}$.

3 Instantons in Commutative Coordinates

3.1 Instantons and their Interpretation in Physics

Instantons are localized finite-action solutions to the equations of motion of a classical theory in the Euclidean space [41]. Thus, they are rather events in space-time than particles. Instanton solutions in Yang-Mills theory were discovered in 1975 by Belavin, Polyakov, Schwarz and Tyupkin (BPST) [6]. The fact that instantons are nonperturbative solutions is important. Whenever the result of perturbation theory vanishes, the first contribution comes from the instantons. These are very small significant effects for small coupling constants and therefore they are hard to observe. Instantons arise as tunneling trajectories in non-abelian gauge theories. They describe a solution to the gauge field equations in which a vacuum at $t = -\infty$ evolves into another vacuum at $t = +\infty$. In the Higgs regime of electroweak theory they may help to understand the baryon number violation at high energies. In QCD, instanton solutions may explain the fluctuation of the gluon field as well as color confinement. For a detailed discussion on this see e.g. [46, 41].

We will describe in a simple example how instantons appear in a theory in order to understand their interpretation as tunneling trajectories. Therefore, let us consider a particle with unit mass in the following potential:

$$V(x) = (x^2 - x_0^2)^2. \tag{3.1}$$

Close to its minima the potential (3.1) becomes the potential of a harmonic oscillator.

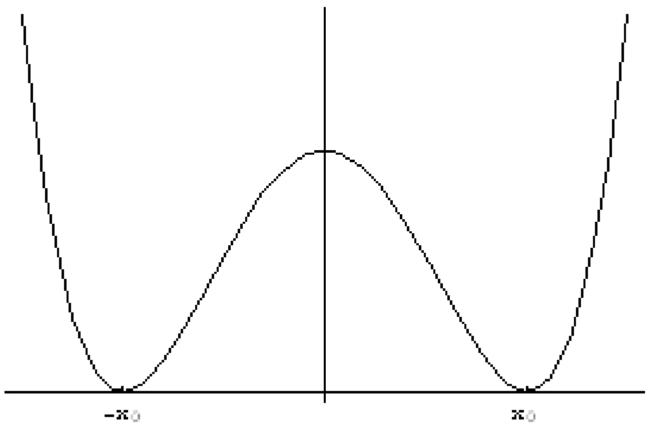

Figure 1: doublewell potential (3.1)

The classical lowest state of the system is degenerated. We have

$$0 = E = \frac{\dot{x}^2}{2} + V(x) \Rightarrow \frac{\dot{x}^2}{2} = -(x^2 - x_0^2)^2. \tag{3.2}$$

The only two solutions to (3.2) are $x(t) = \pm x_0 = \text{const}$, i.e. the particle may be in one well or in the other. The original symmetry of the system with respect to the substitution

$x \mapsto -x$ seems to be broken. From quantum mechanics we know that this is not true. The expectation value of the ground state is zero and there is no degeneracy. Quantum tunneling transfers the wave function from one well to the other. The wave function of the ground state is a superposition of the wave functions in each well.

Now let us consider the above problem in Euclidean coordinates. One can achieve a conversion from Minkowski to Euclidean space-time by a so-called Wick rotation, i.e. substituting

$$t \mapsto \mathrm{i}\tau. \tag{3.3}$$

In our simple example, (3.3) just turns the potential (3.1) into its reflection at the x-axis.

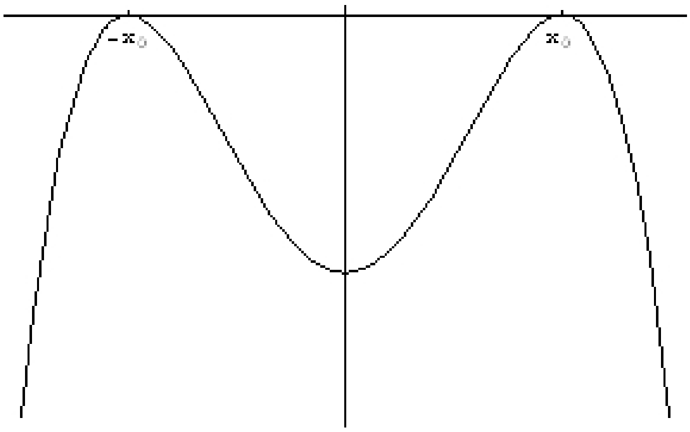

Figure 2: doublehump potential

We are again interested in classical solutions of the lowest state. Now we have

$$0 = E = \frac{\dot{x}^2}{2} - V(x) \Rightarrow \frac{\dot{x}^2}{2} = (x^2 - x_0^2)^2. \tag{3.4}$$

We find an additional solution of the form

$$x(\tau) = x_0 \tanh(\sqrt{2}x_0(\tau - \tau_c)). \tag{3.5}$$

This solution is obtained when the particle slides from one hump $x(-\infty) = -x_0$ and stops on the other $x(+\infty) = x_0$. Such a solution is called instanton. This should clarify why instantons are interpreted as tunneling trajectories. Note that the solution (3.5) is independent of the center at τ_c.

3.2 Instantons in Yang-Mills Theory

We will consider four-dimensional Euclidean pure Yang-Mills theory [39, 41, 14], i.e. we have the space $\mathbb{R}^4$ with the Euclidean metric diag$(1,1,1,1)$. We specify to the gauge group $G = SU(2)$ with its Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$ generated by the three Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.6}$$

We consider the trivial principal bundle $P = \mathbb{R}^4 \times G$. The four components $A_\mu(x) \in \mathfrak{g}$ of a matrix-valued gauge potential are defined on $\mathbb{R}^4$ depending on the global section σ on P. A connection A (the gauge potential) on $\mathbb{R}^4 \times G$ is obtained from $\sigma^* A = A_\mu(x)\ dx^\mu$ (cf. chapter 2).

In pure Yang-Mills theory, i.e. without matter fields, one has the following action:

$$S = -\frac{1}{2g_{YM}^2} \int_{\mathbb{R}^4} \mathrm{tr}(F \wedge *F) = -\frac{1}{2g_{YM}^2} \int_{\mathbb{R}^4} \mathrm{d}^4x\ \mathrm{tr}(F_{\mu\nu}\ F_{\mu\nu}), \tag{3.7}$$

where g_{YM} is a coupling constant and the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ are the components of the curvature F. Here $*$ denotes the Hodge star operator which acts on a r-form Ω (expressed on a chart as $\Omega = \Omega_{\mu_1,\ldots,\mu_r}\ \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_r}$) as

$$*\ \Omega = \frac{1}{(n-r)!}\ \Omega_{\mu_1,\ldots,\mu_r}\ \epsilon_{\mu_1,\ldots,\mu_r,\mu_{r+1},\ldots,\mu_n}\ \mathrm{d}x^{\mu_{r+1}} \wedge \ldots \wedge \mathrm{d}x^{\mu_n}. \tag{3.8}$$

The action (3.7) is invariant under gauge transformations. The Yang-Mills equations of motion,

$$D * F = 0, \tag{3.9}$$

which reads in components

$$D_\mu\ F_{\mu\nu} = \partial_\mu\ F_{\mu\nu} + [A_\mu\ ,\ F_{\mu\nu}] = 0, \tag{3.10}$$

are obtained by extremizing the action (3.7). Here D is the exterior covariant derivative and D_μ is the covariant derivative acting on $F_{\mu\nu}$.

Possible solutions to (3.10) are obtained by demanding the curvature to be self-dual or anti-self-dual, i.e. $*F = \pm F$ and the field strength to satisfy

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\ \epsilon_{\mu\nu\lambda\rho}\ F_{\lambda\rho} = \pm F_{\mu\nu}, \tag{3.11}$$

where $*F = \frac{1}{2}\ \tilde{F}_{\mu\nu}\ \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$. This solves (3.10) since $D_\mu\ \tilde{F}_{\mu\nu} = 0$ due to the Bianchi identity (2.11).

We need to demand that $F_{\mu\nu}$ vanishes at infinity, in order to obtain a finite action (3.7). Therefore, the gauge potential has to be a pure gauge, i.e. some gauge transformation of zero,

$$A_\mu(x) = g(x)\ \partial_\mu(g(x))^{-1} \text{ for } g : \mathbb{R}^4 \to SU(2), \tag{3.12}$$

on the boundary of $\mathbb{R}^4$, which is simply the three-sphere $S^3 = \{x^\mu \mid x^\mu\ x_\mu = r\}$ with radius $r < \infty$. To avoid radial differentiation, one can always gauge away the radial component of A_μ [41]. Since the action is gauge invariant, the transformed A'_μ has to have finite action as well. Then one ends up with the boundary condition

$$A'_\mu(\zeta_1, \zeta_2, \zeta_3) = \tilde{g}(\zeta_1, \zeta_2, \zeta_3)\ \partial_\mu(\tilde{g}(\zeta_1, \zeta_2, \zeta_3))^{-1}, \tag{3.13}$$

where $\tilde{g}(\zeta_1, \zeta_2, \zeta_3) \in SU(2)$ is defined on S^3 and thus depends only on the three coordinates ζ_1, ζ_2 and ζ_3 parameterizing S^3. Any $g \in SU(2)$ can be uniquely written as $g = a^\mu e^\dagger_\mu$, where $\left(e^\dagger_\mu\right) = \left(\mathrm{i}\sigma_a\ ,\ \mathbf{1}_2\right)$ and $a^\mu \in \mathbb{R}^4$ with $a^\mu a_\mu = 1$. Therefore, we have found a map $U : S^3 \to S^3$ which can be divided into a discrete infinity of homotopy classes. From the table (2.18) we know that $\pi_3(SU(2)) \cong \mathbb{Z}$. Thus, the homotopy classes are characterized each by an integer n. This integer shows the number of times $SU(2)$ is wrapped around S^3 and is called the winding number of $SU(2) \cong S^3$ around S^3 (cf. chapter 2).

The same integer appears in another situation as we will see now. The solutions given by (3.11) are in fact absolute minima of the action S for a given number of instantons n. Let us consider the inequality

$$-\frac{1}{4g_{YM}^2}\int_{\mathbb{R}^4} \mathrm{tr}((F-*F)\wedge(F-*F)) = -\frac{1}{4g_{YM}^2}\int_{\mathbb{R}^4} \mathrm{d}^4x\ \mathrm{tr}((F_{\mu\nu}-\tilde{F}_{\mu\nu})\ (F_{\mu\nu}-\tilde{F}_{\mu\nu})) \geq 0$$

which implies

$$S \geq -\frac{1}{2g_{YM}^2}\int_{\mathbb{R}^4} \mathrm{d}^4x\ \mathrm{tr}(F_{\mu\nu}\ \tilde{F}_{\mu\nu}). \tag{3.14}$$

We have used the fact that $\tilde{F}_{\mu\nu}\ \tilde{F}_{\mu\nu} = F_{\mu\nu}\ F_{\mu\nu}$. Obviously, the right hand side in (3.14) is equal to S if $F_{\mu\nu}$ is self-dual. One can define

$$n = -\frac{1}{16\ \pi^2}\int_{\mathbb{R}^4} \mathrm{d}^4x\ \mathrm{tr}(F_{\mu\nu}\ \tilde{F}_{\mu\nu}), \tag{3.15}$$

since after computations of this integral, using Gauss's theorem, we find that it gives an integer, the instanton number. Finally one can show that this integer is equal to the winding number (see e.g. [41, 39]).

3.3 't Hooft Solution and Singular Gauge Transformations

Let us introduce the matrix-valued tensors

$$\eta_{\mu\nu} = \eta^a_{\mu\nu}\ \mathrm{i}\sigma_a \quad \text{and} \quad \bar{\eta}_{\mu\nu} = \bar{\eta}^a_{\mu\nu}\ \mathrm{i}\sigma_a, \tag{3.16}$$

where

$$\eta^a_{\mu\nu} \ = \ \begin{cases} \epsilon^a_{bc} & \text{for} \quad \mu=b\,,\ \nu=c \\ \delta^a_b & \text{for} \quad \mu=b\,,\ \nu=4 \\ -\delta^a_c & \text{for} \quad \mu=4\,,\ \nu=c \end{cases} \tag{3.17}$$

and

$$\bar{\eta}^a_{\mu\nu} \ = \ \begin{cases} \epsilon^a_{bc} & \text{for} \quad \mu=b\,,\ \nu=c \\ -\delta^a_b & \text{for} \quad \mu=b\,,\ \nu=4 \\ \delta^a_c & \text{for} \quad \mu=4\,,\ \nu=c \end{cases} \tag{3.18}$$

are the self-dual ($\frac{1}{2}\ \epsilon_{\mu\nu\lambda\rho}\ \eta_{\lambda\rho} = \eta_{\mu\nu}$) and anti-self-dual ($\frac{1}{2}\ \epsilon_{\mu\nu\lambda\rho}\ \bar{\eta}_{\lambda\rho} = -\bar{\eta}_{\mu\nu}$) 't Hooft tensors respectively [39]. We consider the matrices (which already appeared in section 3.2)

$$\big(e_\mu\big) \ = \ \big(-\mathrm{i}\sigma_a\,,\ \mathbf{1}_2\big) \qquad \text{and} \qquad \big(e^\dagger_\mu\big) \ = \ \big(\mathrm{i}\sigma_a\,,\ \mathbf{1}_2\big). \tag{3.19}$$

They have the following properties:

$$e^\dagger_\mu\ e_\nu = \delta_{\mu\nu}\mathbf{1}_2 + \eta^a_{\mu\nu}\ \mathrm{i}\sigma_a = \delta_{\mu\nu}\mathbf{1}_2 + \eta_{\mu\nu} \tag{3.20}$$

and

$$e_\mu \ e_\nu^\dagger = \delta_{\mu\nu}\mathbf{1}_2 + \bar{\eta}^a_{\mu\nu} \ \mathrm{i}\sigma_a = \delta_{\mu\nu}\mathbf{1}_2 + \bar{\eta}_{\mu\nu}. \tag{3.21}$$

For convenience we introduce the notation

$$\mathbf{x} = x^\mu e_\mu^\dagger \quad \text{and} \quad \mathbf{x}^\dagger = x^\mu e_\mu \tag{3.22}$$

with $\{x^\mu\} \in \mathbb{R}^4$. We will use all these notation throughout the next sections.

The following 't Hooft ansatz is used to obtain a self-dual field strength:

$$A_\mu = \frac{1}{2} \ \bar{\eta}_{\mu\nu}\partial_\nu \ (\ln\phi). \tag{3.23}$$

Here ϕ is some arbitrary scalar function. Substitution of (3.23) into (3.11) yields

$$\frac{1}{\phi}\partial_\mu\partial_\mu \ \phi = 0. \tag{3.24}$$

Hence, we find that with this ansatz (3.23), only singular solution ϕ of equation (3.24) leads to non-trivial solutions for A_μ, namely

$$\phi_n = 1 + \sum_{i=1}^{n} \frac{\Lambda_i^2}{r_i^2}, \tag{3.25}$$

where $r_i^2 = \delta_{\mu\nu} \ (x^\mu - b_i^\mu)(x^\nu - b_i^\nu) = \delta_{\mu\nu} \ x_i^\mu x_i^\mu$ and Λ_i, b_i^μ are some real constants. Inserting (3.25) into (3.23), we obtain

$$A_\mu = -\bar{\eta}_{\mu\nu} \ \frac{1}{1 + \sum_{j=1}^n \frac{\Lambda_j^2}{r_j^2}} \ \sum_{i=1}^{n} \frac{\Lambda_i^2 x_i^\nu}{r_i^4}. \tag{3.26}$$

This solution has n singularities at $x^\mu = b_i^\mu$ for $i = 1, \ldots, n$, which can be removed by a singular gauge transformation [19, 44]. One such function is

$$g_n = \frac{Z_n}{\sqrt{Z_n^\mu \ Z_{n\mu}}} \in SU(2) \qquad \text{with} \quad Z_n = Z_n^\mu \ e_\mu^\dagger = \sum_{i=1}^{n} \beta_i \ \frac{x_i^\mu}{r_i^2} \ e_\mu^\dagger, \tag{3.27}$$

where β_i are some real constants. As it is pointed out in [41, 19], new singularities arise in $A'_\mu = g_n \ A_\mu \ g_n^{-1} + g_n \ \partial_\mu g_n^{-1}$ since g_n has an extra singularity at $Z_n^\mu = 0$. This is always the case unless the b_i^μ are aligned on a straight line [19], when one can choose the β_i in (3.27) in a way such that $Z_n^\mu \neq 0$.

For $n = 1$ and $\beta_1 = 1$ the singular gauge transformation (3.27) simply reduces to

$$g_1 = \frac{\mathbf{x}_1}{\sqrt{r_1^2}} \in SU(2). \tag{3.28}$$

This matrix has no singularity except at $x^\mu = b_1^\mu$. Then the transformed gauge potential becomes

$$A'_\mu = -\eta_{\mu\nu} \ \frac{x_1^\nu}{r_1^2 + \Lambda_1^2}. \tag{3.29}$$

which is nonsingular at finite r. One can calculate the field strength

$$F'_{\mu\nu} = 2\ \eta_{\mu\nu}\ \frac{\Lambda_1^2}{(r_1^2 + \Lambda_1^2)^2} \tag{3.30}$$

which is self-dual since $\eta_{\mu\nu}$ is so.

Furthermore, for $n = 2$, the b_i^μ necessarily lie on a straight line. If one chooses $\beta_1 = -\beta_2 = 1$ in (3.27) then $Z_2^\mu = \frac{x_1^\mu}{r_1^2} - \frac{x_2^\mu}{r_2^2} \neq 0$ except at infinity. Thus, the transformed A'_μ is nonsingular at finite r.

For arbitrary n one can also construct a singular gauge transformation, which removes all singularities to infinity [19, 44], but its explicit expression is complicated. In [19] it is shown that

$$\tilde{g}_n(x^\mu) = \breve{g}_n(x^\mu)\ \breve{g}_{n-1}(x^\mu) \ldots \breve{g}_1(x^\mu) \tag{3.31}$$

generates the transformation $A'_\mu = \tilde{g}_n\ A_\mu\ \tilde{g}_n^{-1} + \tilde{g}_n\ \partial_\mu \tilde{g}_n^{-1}$ which removes all singularities at $x^\mu = b_i^\mu$ in (3.26). Here $\breve{g}_1 = g_1$ and $\breve{g}_i$ is recursively defined by

$$\breve{g}_i(x^\mu) = \frac{\mathbf{x}_i}{\sqrt{r_i^2}}\ \breve{g}_1^{-1}(b_i^\mu)\ \breve{g}_2^{-1}(b_i^\mu) \ldots \breve{g}_{i-1}^{-1}(b_i^\mu). \tag{3.32}$$

As an example for $n = 2$ we have

$$\tilde{g}_2 = \frac{\mathbf{x}_2\ (\mathbf{b}_2 - \mathbf{b}_1)^\dagger\ \mathbf{x}_1}{\sqrt{r_1^2 r_2^2 (b_2 - b_1)^2}}. \tag{3.33}$$

Remarks. Note that neither the 't Hooft nor the transformed solutions are all instanton solutions for a given $n > 1$. The moduli space of a general n-instanton solution [3] is equal to $8n - 3$.

To be precise, the transformations generated by (3.27) and (3.31) are not gauge transformations since they are not defined on the whole space $\mathbb{R}^4$. In fact, g_1 as in (3.28) is the transition function relating $A = A_\mu\ dx^\mu$ and $A' = A'_\mu\ dx^\mu$, where

$$A_\mu = -\bar{\eta}_{\mu\nu}\ \frac{x_1^\nu}{r_1^2(r_1^2 + \Lambda_1^2)} \qquad \text{and} \qquad A'_\mu = -\eta_{\mu\nu}\ \frac{x_1^\nu}{r_1^2 + \Lambda_1^2}$$

as above. A and A' are defined on two patches covering the four-sphere, A on $S^4\backslash\{0\}$ and A' on $S^4\backslash\{\infty\}$ respectively (see e.g. [42]). They are local representatives of a connection on a principal bundle over S^4 with the gauge group $G = SU(2)$ (see chapter 2).

3.4 ADHM Construction for $SU(2)$ Instantons

The Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction [2] is a method of how to obtain in principle all instanton solutions in a systematic way and is based on a duality, i.e. a one-to-one correspondence between the instanton moduli space and the moduli space of ADHM data [21]. We will verify just one direction of this duality, namely that starting with some ADHM data this construction will give a self-dual field strength. The mathematical framework of the

ADHM construction is twistor theory. The ADHM construction of an n-instanton solution in $SU(2)$ is based on the ADHM data, a $(2n+2)\times 2n$ matrix $\Delta = I + J\ (\mathbf{x}\otimes \mathbf{1}_n)$, where I and J are constant $(2n+2)\times 2n$ matrices and $\mathbf{x} = x^\mu e_\mu^\dagger$ as usual. This matrix has to satisfy the following two conditions (see e.g. [9, 31] and references therein):

$$\Delta^\dagger\Delta \quad \text{is invertible} \tag{3.34}$$

and

$$[\Delta^\dagger\Delta, e_\mu\otimes \mathbf{1}_n] = 0 \qquad \forall x. \tag{3.35}$$

It is not difficult to see that these two conditions (3.34) and (3.35) are fulfilled if we demand that Δ satisfies the factorization condition

$$\Delta^\dagger\Delta = \mathbf{1}_2\otimes f^{-1}_{n\times n}. \tag{3.36}$$

Then we look for some specific zero mode of $\Delta^\dagger$, i.e.

$$\Delta^\dagger\mathbf{\Psi} = 0, \tag{3.37}$$

where $\mathbf{\Psi}$ is a $(2n+2)\times 2$ matrix. This zero-mode eigenfunction $\mathbf{\Psi}$ should be normalized, i.e.

$$\mathbf{\Psi}^\dagger\mathbf{\Psi} = \mathbf{1}_2. \tag{3.38}$$

For Δ and $\mathbf{\Psi}$ satisfying (3.34)-(3.38) the gauge potential may be chosen in the form

$$A = \mathbf{\Psi}^\dagger\ \mathrm{d}\mathbf{\Psi} \tag{3.39}$$

with the components

$$A_\mu = \mathbf{\Psi}^\dagger\ \partial_\mu\mathbf{\Psi}. \tag{3.40}$$

Then the resulting field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ will be self-dual if Δ and $\mathbf{\Psi}$ obey the completeness relation

$$\mathbf{\Psi}\ \mathbf{\Psi}^\dagger\ + \Delta\,(\Delta^\dagger\Delta)^{-1}\Delta^\dagger = \mathbf{1}_{2n+2}. \tag{3.41}$$

From (3.38) we obtain

$$0 = \partial_\mu(\mathbf{\Psi}^\dagger\mathbf{\Psi}) = \partial_\mu(\mathbf{\Psi}^\dagger)\ \mathbf{\Psi} + \mathbf{\Psi}^\dagger\ (\partial_\mu\mathbf{\Psi}) \tag{3.42}$$

and from (3.37)

$$0 = \partial_\mu(\Delta^\dagger\mathbf{\Psi}) = (\partial_\mu\Delta^\dagger)\mathbf{\Psi} + \Delta^\dagger(\partial_\mu\mathbf{\Psi}). \tag{3.43}$$

Thus, using (3.36), (3.41), (3.42) and (3.43), we find

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu(\mathbf{\Psi}^\dagger\partial_\nu\mathbf{\Psi}) - \partial_\nu(\mathbf{\Psi}^\dagger\partial_\mu\mathbf{\Psi}) + [\mathbf{\Psi}^\dagger(\partial_\mu\mathbf{\Psi}),\ \mathbf{\Psi}^\dagger(\partial_\nu\mathbf{\Psi})] \\ &= (\partial_\mu\mathbf{\Psi}^\dagger)\ (\partial_\nu\mathbf{\Psi}) + \mathbf{\Psi}^\dagger\ (\partial_\mu\partial_\nu\mathbf{\Psi}^\dagger) - (\partial_\nu\mathbf{\Psi}^\dagger)\ (\partial_\mu\mathbf{\Psi}) - \mathbf{\Psi}^\dagger\ (\partial_\nu\partial_\mu\mathbf{\Psi}^\dagger) \\ &+ \mathbf{\Psi}^\dagger(\partial_\mu\mathbf{\Psi})\ \mathbf{\Psi}^\dagger(\partial_\nu\mathbf{\Psi}) - \mathbf{\Psi}^\dagger(\partial_\nu\mathbf{\Psi})\ \mathbf{\Psi}^\dagger(\partial_\mu\mathbf{\Psi}) \\ &= (\partial_\mu\mathbf{\Psi}^\dagger)\ (\mathbf{1}_{2n+2} - \mathbf{\Psi}\ \mathbf{\Psi}^\dagger)\ (\partial_\nu\mathbf{\Psi}) - (\partial_\nu\mathbf{\Psi}^\dagger)\ (\mathbf{1}_{2n+2} - \mathbf{\Psi}\ \mathbf{\Psi}^\dagger)\ (\partial_\mu\mathbf{\Psi}) \\ &= (\partial_\mu\mathbf{\Psi}^\dagger)\ (\Delta\,(\Delta^\dagger\Delta)^{-1}\Delta^\dagger)\ (\partial_\nu\mathbf{\Psi}) - (\partial_\nu\mathbf{\Psi}^\dagger)\ (\Delta\,(\Delta^\dagger\Delta)^{-1}\Delta^\dagger)\ (\partial_\mu\mathbf{\Psi}) \\ &= \mathbf{\Psi}^\dagger\ (\partial_\mu\Delta)\ (\Delta^\dagger\Delta)^{-1}(\partial_\nu\Delta^\dagger)\ \mathbf{\Psi} - \mathbf{\Psi}^\dagger\ (\partial_\nu\Delta)\ (\Delta^\dagger\Delta)^{-1}(\partial_\mu\Delta^\dagger)\ \mathbf{\Psi} \\ &= \mathbf{\Psi}^\dagger\ J\ e_\mu^\dagger\ (\Delta^\dagger\Delta)^{-1}\ e_\nu\ J^\dagger\ \mathbf{\Psi} - \mathbf{\Psi}^\dagger\ J\ e_\nu^\dagger\ (\Delta^\dagger\Delta)^{-1}\ e_\mu\ J^\dagger\ \mathbf{\Psi} \\ &= 2\ \mathbf{\Psi}^\dagger\ J\ (\Delta^\dagger\Delta)^{-1}\ \eta_{\mu\nu}\ J^\dagger\ \mathbf{\Psi}, \end{aligned} \tag{3.44}$$

where $\eta_{\mu\nu}$ is the self-dual 't Hooft tensor (3.17). Hence, the anti-self-dual part of $F_{\mu\nu}$ in (3.44) is zero.

Note that this construction can be easily generalized to the case with the gauge group $SU(N)$.

3.5 Nonsingular Multi-Instantons

In this section we will establish another approach of how to obtain multi-instanton solutions which are regular at finite points. Instead of finding transformations, which remove singularities from singular solutions, we will construct solutions which are nonsingular at finite points from the very beginning. Here we will use the ADHM construction (cf. section 3.4).

To find n-instanton solutions via ADHM construction let us take the following data [31]:

$$\Delta = \begin{pmatrix} \Lambda_1 \mathbf{1}_2 & \dots & \Lambda_n \mathbf{1}_2 \\ \mathbf{x}_1 & & \mathbf{0}_2 \\ & \ddots & \\ \mathbf{0}_2 & & \mathbf{x}_n \end{pmatrix} = \begin{pmatrix} \Lambda_1 \mathbf{1}_2 & \dots & \Lambda_n \mathbf{1}_2 \\ -\mathbf{b}_1 & & \mathbf{0}_2 \\ & \ddots & \\ \mathbf{0}_2 & & -\mathbf{b}_n \end{pmatrix} + \begin{pmatrix} \mathbf{0}_2 & \dots & \mathbf{0}_2 \\ \mathbf{1}_2 & & \mathbf{0}_2 \\ & \ddots & \\ \mathbf{0}_2 & & \mathbf{1}_2 \end{pmatrix} \begin{pmatrix} \mathbf{x} & & \mathbf{0}_2 \\ & \ddots & \\ \mathbf{0}_2 & & \mathbf{x} \end{pmatrix} \tag{3.45}$$

with $\mathbf{x}_i = \mathbf{x} - \mathbf{b}_i = (x^\mu - b_i^\mu)\ e_\mu^\dagger$ as usual. We obtain

$$\Delta^\dagger \Delta = \mathbf{1}_2 \otimes (\delta_{ij} r_j^2 + \Lambda_i \Lambda_j) = \mathbf{1}_2 \otimes (R + \Lambda \Lambda^T), \tag{3.46}$$

where

$$R \ = \ \begin{pmatrix} r_1^2 & & 0 \\ & \ddots & \\ 0 & & r_n^2 \end{pmatrix} \quad \text{and} \quad \Lambda^T \ = \ (\Lambda_1, \dots, \Lambda_n). \tag{3.47}$$

From (3.46) we see that condition (3.36) is satisfied. Thus, also (3.34) and (3.35) are satisfied. Indeed, by direct calculation we find

$$\mathbf{1}_2 \otimes f_{n\times n} = (\Delta^\dagger\ \Delta)^{-1} \ = \ \mathbf{1}_2 \otimes \left(R^{-1} - R^{-1}\Lambda\ \phi_n^{-1} \Lambda^T R^{-1} \right) \tag{3.48}$$

with

$$\phi_n = 1 + \sum_{i=1}^{n} \frac{\Lambda_i^2}{r_i^2} = 1 + \Lambda^T R^{-1} \Lambda \tag{3.49}$$

as in (3.25). Furthermore, we write $\mathbf{\Psi}$ as the column vector

$$\mathbf{\Psi} = \begin{pmatrix} \Psi_0 \\ \Psi_1 \\ \vdots \\ \Psi_n \end{pmatrix}, \tag{3.50}$$

where the Ψ_i for $i = 0, 1, \dots, n$ are 2×2 matrices.

Finally we just need to solve the following two equations in order to obtain instanton solutions from the data (3.45) [1] :

$$\Lambda_i \ \Psi_0 + \mathbf{x}_i^\dagger \ \Psi_i = \mathbf{0}_2 \qquad i = 1, \ldots, n \tag{3.51}$$

and

$$\Psi_0^\dagger \ \Psi_0 + \Psi_1^\dagger \ \Psi_1 + \ldots + \Psi_n^\dagger \ \Psi_n = \mathbf{1}_2. \tag{3.52}$$

Then we can calculate the gauge potential using (3.39) and the field strength using (3.44).

't Hooft ansatz. As a remark, we want to mention that the 't Hooft ansatz now reads [31]

$$\Psi_0^{\text{'t Hooft}} = \phi_n^{-\frac{1}{2}} \ \mathbf{1}_2 \quad \text{and} \quad \Psi_i^{\text{'t Hooft}} = -\mathbf{x}_i \ \frac{\Lambda_i}{r_i^2} \ \phi_n^{-\frac{1}{2}}, \tag{3.53}$$

for $i = 1, \ldots, n$. This ansatz produces a singular solution. Singularities at $x^\mu = b_i^\mu$ have to be removed by singular gauge transformations as described in section 3.3. In this section we want to obtain instantons by a nonsingular ansatz.

One instanton. We start with [18]

$$\Psi_0^{\text{regular}} = \mathbf{x}_1^\dagger \ f_1 \qquad \text{and} \qquad \Psi_1^{\text{regular}} = -\Lambda_1 \ f_1, \tag{3.54}$$

which solves (3.51). Then from (3.52) one obtains

$$f_1 = \frac{1}{\sqrt{r_1^2 + \Lambda_1^2}} \ \mathbf{1}_2.$$

This solution $\mathbf{\Psi}^{\text{regular}}$ (3.54) is obviously nonsingular at finite r. Then from (3.39) one can calculate

$$A_\mu = \Psi_0^\dagger \ \partial_\mu \Psi_0 + \Psi_1^\dagger \ \partial_\mu \Psi_1 = -\eta_{\mu\nu} \ \frac{x_1^\nu}{r_1^2 + \Lambda_1^2}. \tag{3.55}$$

This coincides with the calculation in section 3.3 for one instanton in nonsingular gauge (3.29) producing the BPST solution. Note that the singular gauge transformation (performed in section 3.3) from the singular to the regular gauge now reads $\mathbf{\Psi}^{\text{regular}} = \mathbf{\Psi}^{\text{'t Hooft}} \ g_1^\dagger$ with g_1 as in (3.28).

Two instantons. In this case one can consider the ansatz

$$\begin{aligned} \Psi_0^{\text{regular}} &= \mathbf{x}_1^\dagger \ (\mathbf{b}_2 - \mathbf{b}_1) \ \mathbf{x}_2^\dagger \ f_2 = \mathbf{x}_2^\dagger \ (\mathbf{b}_2 - \mathbf{b}_1) \ \mathbf{x}_1^\dagger \ f_2, \\ \Psi_1^{\text{regular}} &= -\Lambda_1 \ (\mathbf{b}_2 - \mathbf{b}_1) \ \mathbf{x}_2^\dagger \ f_2 \qquad \text{and} \qquad \Psi_2^{\text{regular}} = -\Lambda_2 \ (\mathbf{b}_2 - \mathbf{b}_1) \ \mathbf{x}_1^\dagger \ f_2, \end{aligned} \tag{3.56}$$

which solves again (3.51). From (3.52) one obtains the normalization factor given by

$$f_2 = \frac{1}{\sqrt{(b_2 - b_1)^2 \ (r_1^2 \ r_2^2 + \Lambda_1^2 \ r_2^2 + \Lambda_2^2 \ r_1^2)}} \ \mathbf{1}_2.$$

This is also nonsingular at finite r since r_1^2 and r_2^2 cannot be zero simultaneously.

[1] The completeness relation (3.41) in the commutative case is automatically satisfied.

We want to make the following remark: if one transforms again $\mathbf{\Psi}^{\text{'t Hooft}} \mapsto \mathbf{\Psi}^{\text{'t Hooft}}\, \tilde{g}_2^\dagger(x)$ with $\mathbf{\Psi}^{\text{'t Hooft}}$ as in (3.53) for $n=2$ and $\tilde{g}_2(x)$ as in (3.33), then one obtains exactly the regular ansatz (3.56).

Three instantons. In order to construct three instantons one can use the ansatz

$$\begin{aligned}
\Psi_0^{\text{regular}} &= \mathbf{x}_{12}^\dagger \left((r_1^2 - r_2^2)\ \mathbf{x}_3 - \mathbf{x}_{12}\right)\ \mathbf{x}_3^\dagger\ f_3, \\
\Psi_1^{\text{regular}} &= -\Lambda_1\ (\mathbf{b}_2 - \mathbf{b}_1)\ \mathbf{x}_2^\dagger\ \left((r_1^2 - r_2^2)\ \mathbf{x}_3 - \mathbf{x}_{12}\right)\ \mathbf{x}_3^\dagger\ f_3, \\
\Psi_2^{\text{regular}} &= -\Lambda_2\ (\mathbf{b}_2 - \mathbf{b}_1)\ \mathbf{x}_1^\dagger\ \left((r_1^2 - r_2^2)\ \mathbf{x}_3 - \mathbf{x}_{12}\right)\ \mathbf{x}_3^\dagger\ f_3 \text{ and} \\
\Psi_3^{\text{regular}} &= -\Lambda_3\ \left((r_1^2 - r_2^2)\ \mathbf{x}_3 - \mathbf{x}_{12}\right)\ \mathbf{x}_{12}^\dagger\ f_3,
\end{aligned}$$

where $\mathbf{x}_{12} = \mathbf{x}_1\ (\mathbf{b}_2^\dagger - \mathbf{b}_1^\dagger)\ \mathbf{x}_2$. This solves again (3.51). From (3.52) one obtains

$$f_3 = \frac{1}{\sqrt{(b_2-b_1)^2\ d_3^2\ (r_1^2\ r_2^2\ r_3^2 + \Lambda_1^2\ r_2^2\ r_3^2 + \Lambda_2^2\ r_1^2\ r_3^2 + \Lambda_3^2\ r_1^2\ r_2^2)}}\mathbf{1}_2.$$

Here $d_3^2 = (r_1^2\ r_2^2\ (b_2-b_1)^2 + (r_1^2 - r_2^2)\ (r_1^2\ (b_3-b_2)^2 - r_2^2\ (b_3-b_1)^2))$ is non-zero for any x^μ (see general case below). Therefore, we find that f_3 as well as $\mathbf{\Psi}^{\text{regular}}$ are again nonsingular at finite points. The transformation $\mathbf{\Psi}^{\text{'t Hooft}} \mapsto \mathbf{\Psi}^{\text{'t Hooft}}\, \tilde{g}_3^\dagger(x)$ with $\mathbf{\Psi}^{\text{'t Hooft}}$ as in (3.53) and $\tilde{g}_3(x)$ as in (3.31) gives a regular solution but is not identically equal to $\mathbf{\Psi}^{\text{regular}}$.

Recursion formula for n instantons. In order to construct a nonsingular n-instanton solution for $n \geq 2$ one may consider the following ansatz:

$$\begin{aligned}
\Psi_0^{\text{regular}} &= \ \mathbf{x}_{1\ldots n}^\dagger\ f_n, \\
\Psi_1^{\text{regular}} &= -\Lambda_1\ \mathbf{q}_2\ \mathbf{x}_2^\dagger\ \mathbf{q}_3\ \mathbf{x}_3^\dagger \ldots \mathbf{q}_n\ \mathbf{x}_n^\dagger\ f_n, \\
\Psi_2^{\text{regular}} &= -\Lambda_2\ \mathbf{q}_2\ \mathbf{x}_1^\dagger\ \mathbf{q}_3\ \mathbf{x}_3^\dagger \ldots \mathbf{q}_n\ \mathbf{x}_n^\dagger\ f_n, \\
\Psi_3^{\text{regular}} &= -\Lambda_3\ \mathbf{q}_3\ \mathbf{x}_1^\dagger\ \mathbf{q}_2\ \mathbf{x}_2^\dagger \ldots \mathbf{q}_n\ \mathbf{x}_n^\dagger\ f_n, \\
&\ \vdots \\
\Psi_n^{\text{regular}} &= -\Lambda_n\ \mathbf{q}_n\ \mathbf{x}_{1\ldots n-1}^\dagger\ f_n.
\end{aligned} \tag{3.57}$$

Here $\mathbf{q}_2 = \mathbf{b_2} - \mathbf{b_1},\ \mathbf{x}_{12} = \mathbf{x}_1 \mathbf{q}_2^\dagger \mathbf{x}_2,\ h_3 = r_1^2 - r_2^2$ and $\mathbf{q}_n,\ \mathbf{x}_{1\ldots n},\ h_{n+1}$ are recursively defined for $n > 2$ by

$$\begin{aligned}
\mathbf{q}_n &= h_n\ \mathbf{x}_n - \mathbf{x}_{1\ldots n-1}, \\
\mathbf{x}_{1\ldots n} &= \mathbf{x}_{1\ldots n-1}\ \mathbf{q}_n^\dagger\ \mathbf{x}_n \quad \text{and} \\
h_{n+1} &= \ h_n^2\ r_n^2 - r_1^2 \ldots r_{n-1}^2\ q_2^2 \ldots q_{n-1}^2.
\end{aligned} \tag{3.58}$$

This solves (3.51). From (3.52) one obtains that the norm of $\mathbf{\Psi}^{\text{regular}}$ will be unity if

$$f_n^{-2} = q_2^2 \ldots q_n^2 (r_1^2 \ldots r_n^2 + \Lambda_1^2 r_2^2 \ldots r_n^2 + \Lambda_2^2 r_1^2 r_3^2 \ldots r_n^2 + \ldots + \Lambda_n^2 r_1^2 \ldots r_{n-1}^2)\mathbf{1}_2 \tag{3.59}$$

The bracket in equation (3.59) is non-zero because the terms in the bracket cannot be zero simultaneously. To show that this solution (3.57) is nonsingular at finite points, one only needs to show that q_n^2 is non-zero for finite x^μ and all $n \geq 2$.

In preparation for this one may show by induction that for $n \geq 2$ we have

$$\mathbf{x}_{1\ldots n} = \mathbf{x}_1^\dagger \; \mathbf{q}_2 \; \mathbf{x}_2^\dagger \; \mathbf{q}_3 \; \mathbf{x}_3^\dagger \ldots \mathbf{q}_n \; \mathbf{x}_n^\dagger = h_{n+1} \; \mathbf{x} - \mathbf{b}_{1\ldots n}, \tag{3.60}$$

where $\mathbf{b}_{1\ldots n}$ depends on $r_1^2, \ldots, r_n^2$ and $\mathbf{b}_1, \ldots, \mathbf{b}_n$ only. Obviously for $n = 2$ one has $\mathbf{x}_{12} = \mathbf{x}_1 \; \mathbf{q}_2^\dagger \; \mathbf{x}_2 = r_1^2 \; \mathbf{x}_2 - \mathbf{x}_1 \; r_2^2 = (r_1^2 - r_2^2) \; \mathbf{x} - (r_1^2 \; \mathbf{b}_2 - r_2^2 \; \mathbf{b}_1) = h_3 \; \mathbf{x} - \mathbf{b}_{12}$. Suppose (3.60) holds for n. With (3.58) and $r_{1\ldots n}^2 = \mathbf{x}_{1\ldots n}^\dagger \; \mathbf{x}_{1\ldots n}$ we find

$$\begin{aligned}
\mathbf{x}_{1\ldots n+1} &= \mathbf{x}_{1\ldots n} \; \mathbf{q}_{n+1}^\dagger \; \mathbf{x}_{n+1} \\
&= h_{n+1} \; \mathbf{x}_{1\ldots n} \; r_{n+1}^2 - r_{1\ldots n}^2 \; \mathbf{x}_{n+1} \\
&= h_{n+1} \; (h_{n+1} \; \mathbf{x} - \mathbf{b}_{1\ldots n}) \; r_{n+1}^2 - r_{1\ldots n}^2 \; (\mathbf{x} - \mathbf{b}_{n+1}) \\
&= (h_{n+1}^2 \; r_{n+1}^2 - r_{1\ldots n}^2) \; \mathbf{x} - (h_{n+1} \; \mathbf{b}_{1\ldots n} \; r_{n+1}^2 - r_{1\ldots n}^2 \; \mathbf{b}_{n+1}) \\
&= h_{n+2} \; \mathbf{x} - \mathbf{b}_{\mathbf{1\ldots n+1}}
\end{aligned}$$

with $\mathbf{b}_{1\ldots n+1} = h_{n+1} \; \mathbf{b}_{1\ldots n} \; r_{n+1}^2 - r_{1\ldots n}^2 \; \mathbf{b}_{n+1}$. This only depends on $r_1^2, \ldots, r_{n+1}^2$ and $\mathbf{b}_1, \ldots, \mathbf{b}_{n+1}$. □

From (3.60) it follows that

$$\mathbf{q}_n = h_n \; \mathbf{x}_n - \mathbf{x}_{1\ldots n-1} = h_n \; (\mathbf{x} - \mathbf{b}_n) - (h_n \; \mathbf{x} - \mathbf{b}_{1\ldots n-1}) = \mathbf{b}_{1\ldots n-1} - h_n \; \mathbf{b}_n \tag{3.61}$$

for $n > 2$. This is why we have chosen $h_3 = r_1^2 - r_2^2$ and $h_n = h_{n-1}^2 \; r_{n-1}^2 - r_{1\ldots n-2}^2$ for $n \geq 3$.

Secondly, we need to show that the following applies for $n > 2$:

$$h_n = 1 \Rightarrow \mathbf{b}_{1\ldots n-1} \neq \mathbf{b} \qquad \text{for some constant matrix } \mathbf{b} \neq \mathbf{b}_i, \; i = 1, \ldots, n-1. \tag{3.62}$$

We have $h_3 = r_1^2 - r_2^2 = 1 \Rightarrow r_1^2 = r_2^2 + 1$. Suppose $\mathbf{b}_{12} = \mathbf{b}$ with $\mathbf{b} \neq \mathbf{b}_i, \; i = 1, 2$. We can always choose a frame where $(\mathbf{b} - \mathbf{b}_2)$ is invertible. Then we find $r_2^2 \; (\mathbf{b}_2 - \mathbf{b}_1)(\mathbf{b} - \mathbf{b}_2)^{-1} = \mathbf{1}_2$ which is a contradiction. Therefore (3.62) holds for $n = 3$. Now suppose $h_n = 1 \Rightarrow \mathbf{b}_{1\ldots n-1} \neq \mathbf{b}$ holds for a given n where $\mathbf{b} \neq \mathbf{b}_i$ for $i = 1, \ldots, n-1$. Let $h_{n+1} = 1$ and suppose $\mathbf{b}_{1\ldots n} = \mathbf{b}$ and $\mathbf{b} \neq \mathbf{b}_i$ for $i = 1, \ldots, n$. We obtain

$$\begin{aligned}
h_n \; r_n^2 \; \mathbf{b}_{1\ldots n-1} - r_{1\ldots n-1}^2 \; \mathbf{b}_n &= \mathbf{b} \\
\Rightarrow h_n \; r_n^2 \; \mathbf{b}_{1\ldots n-1} - (h_n^2 \; r_n^2 - h_{n+1}) \; \mathbf{b}_n &= \mathbf{b} \\
\Rightarrow h_n \; r_n^2 (\mathbf{b}_{1\ldots n-1} - h_n \; \mathbf{b}_n) \; (\mathbf{b} - \mathbf{b}_n)^{-1} &= \mathbf{1}_2.
\end{aligned}$$

This is a contradiction since $(\mathbf{b} - \mathbf{b}_n) \neq (\mathbf{b}_{1\ldots n-1} - \mathbf{b}_n)$ for some $\mathbf{b} \neq \mathbf{b}_i, \; i = 1, \ldots, n-1$ due to our assumption. □

Furthermore, we should show that the following applies for $n > 2$ and $\mathbf{q}_i \neq 0$ with $i = 1, \ldots, n-1$:

$$\text{if } r_i^2 = 0 \text{ for one } i \in \{1, \ldots, n-1\} \text{ then } h_n \neq 0. \tag{3.63}$$

Therefore, consider $n = 3$ and $r_i^2 = 0$ for one $i \in \{1, 2\}$. Then $h_3 = r_1^2 - r_2^2 \neq 0$. Now suppose (3.63) holds for a given n. Consider $r_i^2 = 0$ for one $i = \{1, \ldots, n\}$. Then we find with (3.58) $h_{n+1} = h_n^2 \; r_n^2 - r_1^2 \ldots r_{n-1}^2 \; q_2^2 \ldots q_{n-1}^2 \neq 0$. □

Finally, together with (3.62) and (3.63) we should be able to show by induction that $\mathbf{q}_n$ is non-zero for all $n \geq 2$. For $n = 2$ we have $\mathbf{q}_2 = \mathbf{b}_2 - \mathbf{b}_1$, which is obviously non-zero. For $n = 3$ we have $\mathbf{q}_3 = \mathbf{b}_{12} - h_3 \; \mathbf{b}_3 = r_1^2 \; (\mathbf{b}_2 - \mathbf{b}_3) - r_2^2 \; (\mathbf{b}_1 - \mathbf{b}_3)$. Suppose $\mathbf{q}_3 = 0$. We can always choose a frame where $(\mathbf{b}_1 - \mathbf{b}_3)$ is invertible. Then we find $r_1^2\ (\mathbf{b}_2 - \mathbf{b}_3)\ (\mathbf{b}_1 - \mathbf{b}_3)^{-1} = r_2^2\ \mathbf{1}_2$. This is a contradiction. Consider now $n > 3$. With (3.58) - (3.61) we have

$$
\begin{aligned}
\mathbf{q}_n &= h_n\ \mathbf{x}_n - \mathbf{x}_{1\ldots(n-2)}\ \mathbf{q}_{n-1}^\dagger\ \mathbf{x}_{n-1} \\
&= (h_{n-1}^2\ r_{n-1}^2 - r_{1\ldots(n-2)}^2)\ \mathbf{x}_n - \mathbf{x}_{1\ldots(n-2)}\ (h_{n-1}\ \mathbf{x}_{n-1}^\dagger - \mathbf{x}_{1\ldots(n-2)}^\dagger)\ \mathbf{x}_{n-1} \\
&= h_{n-1}\ r_{n-1}^2\ (h_{n-1}\ \mathbf{x}_n - \mathbf{x}_{1\ldots n-2}) - r_{1\ldots n-2}^2(\mathbf{x}_n - \mathbf{x}_{n-1}) \\
&= h_{n-1}\ r_{n-1}^2\ (\mathbf{b}_{1\ldots n-2} - h_{n-1}\ \mathbf{b}_n) - r_{1\ldots n-2}^2(\mathbf{b}_{n-1} - \mathbf{b}_n)
\end{aligned}
$$

Now suppose that $\mathbf{q}_i \neq 0$ holds for $i = 1, \ldots, n-1$ and suppose $\mathbf{q}_n = 0$.

$$\Rightarrow h_{n-1}\ r_{n-1}^2\ (\mathbf{b}_{1\ldots n-2} - h_{n-1}\ \mathbf{b}_n)\ (\mathbf{b}_{n-1} - \mathbf{b}_n)^{-1} = r_1^2 \ldots r_{n-2}^2\ \mathbf{q}_2^2 \ldots \mathbf{q}_{n-2}^2\ \mathbf{1}_2.$$

This leads to a contradiction because $(\mathbf{b}_{1\ldots n-2} - h_{n-1}\ \mathbf{b}_n) \neq (\mathbf{b}_{n-1} - \mathbf{b}_n)$ from (3.62) and if $r_{n-1}^2 = 0$, then $r_i^2 \neq 0$ for $i = 1, \ldots n-1$. And if one has $r_i^2 = 0$ for $i = 1, \ldots, n-2$, then $r_{n-1} \neq 0$ and $h_{n-1} \neq 0$ (3.63). $\square$

Thus, f_n in (3.59) is nonsingular at finite points. Hence we have found nonsingular n-instanton solutions with (3.57).

The method presented in section 3.3, namely starting with the singular 't Hooft ansatz and transforming the singular solutions, yields by construction regular solutions. Proving that the solutions constructed in this section are regular is not trivial, but they have an important feature: Ψ_0^{regular} is symmetric in $\mathbf{x}_i$.

4 Instantons in Noncommutative Coordinates

4.1 Instantons in String Theory

Instantons have a clear interpretation as a D0-D4 brane system in type II A string theory [55, 21]. Here the number of D0-branes and D4-branes correspond to the instanton number n and the rank of the gauge group N. The ADHM construction is embedded in this D-brane system.

When a constant B-field is present in the background, the D0-D4 brane system becomes stable. This situation is explained by noncommutative instantons. It is well known that studying noncommutative instantons will reveal some aspects of D-brane dynamics [22].

4.2 Noncommutative $\mathbb{R}^4_\theta$

We want to introduce the noncommutative Euclidean space $\mathbb{R}^4_\theta$, i.e. the Euclidean space $\mathbb{R}^4$ where the coordinates $x^\mu \in \mathbb{R}^4$ fail to commute

$$[x^\mu, x^\nu] = \mathrm{i}\ \theta^{\mu\nu}. \tag{4.1}$$

We will consider the simplest case, where $\theta^{\mu\nu}$ does not depend on the coordinates but is a constant antisymmetric tensor. We need to deform the algebra of continuous function endowed with the ordinary product $C^\infty(\mathbb{R}^4, \cdot)$ to an algebra $C^\infty_\theta(\mathbb{R}^4, \star)$ with some associative product $\star$ which gives rise to the non-commutativity (4.1). This is known as the Moyal algebra. Then one can start with a conventional field theory, which has some gauge symmetry, and define a noncommutative version of this field theory. For more details on this see e.g. [16, 34, 49]. Even though the $\mathbb{R}^4_\theta$ is not Lorentz invariant and somehow non-localized, noncommutative gauge theories are important since they arise as a certain limit in string theory [45]. The discussion in section 3.2 is formally unchanged on $\mathbb{R}^4_\theta$, but we are not allowed to use $SU(2)$ as the gauge group since the Moyal commutator does not close in this case. Instead of $SU(2)$ we will consider $U(2)$.

A realization of the product $\star$ is the Moyal product of two functions $f(x)$ and $g(x)$ given by

$$\begin{aligned}(f \star g)(x) &= f(x)\ \exp(\frac{\mathrm{i}}{2}\overleftarrow{\partial}_\mu\, \theta^{\mu\nu}\, \overrightarrow{\partial}_\nu)\ g(x) \\ &= f(x)\cdot g(x) + \frac{\mathrm{i}}{2}\theta^{\mu\nu}(\partial_\mu f(x))(\partial_\nu g(x)) + \dots \end{aligned} \tag{4.2}$$

with a constant antisymmetric tensor $\theta^{\mu\nu}$. Then we obtain for the coordinates the commutation relation (4.1)

$$[x^\mu, x^\nu]_\star = x^\mu \star x^\nu - x^\nu \star x^\mu = \mathrm{i}\ \theta^{\mu\nu}. \tag{4.3}$$

Note that one can always choose a basis where $\theta^{\mu\nu}$ has the canonical form

$$(\theta^{\mu\nu}) = \begin{pmatrix} 0 & \theta^{12} & 0 & 0 \\ -\theta^{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta^{34} \\ 0 & 0 & -\theta^{34} & 0 \end{pmatrix} \tag{4.4}$$

with $\theta^{12} > 0$ and $\theta^{12} + \theta^{34} \geq 0$.

There is another formulation for $\mathbb{R}^4_\theta$, which we will use for our explicit calculations. Let us introduce complex coordinates

$$z^1 = x^1 + \mathrm{i}x^2 \qquad \text{and} \qquad z^2 = x^3 - \mathrm{i}x^4. \tag{4.5}$$

with $x^\mu \in \mathbb{R}^4_\theta$. We choose $\theta^{\mu\nu}$ as in (4.4) and find the only non-vanishing commutators

$$[z^1, \bar{z}^1] = 2\theta^{12} \qquad \text{and} \qquad [z^2, \bar{z}^2] = -2\theta^{34}. \tag{4.6}$$

One now regards $\{\bar{z}^1, z^1\}$ and $\{z^2, \bar{z}^2\}$ as two sets consisting of a creation and a annihilation operator each, denoted by $\{\hat{\bar{z}}^1, \hat{z}^1\}$ and $\{\hat{z}^2, \hat{\bar{z}}^2\}$, which act on a two-oscillator Fock space $\mathcal{H}$ with the basis

$$|n_1, n_2\rangle = \frac{1}{\sqrt{n_1!\; n_2!\; (2\theta^{12})^{n_1}\; (2\theta^{34})^{n_2}}}\,(\hat{\bar{z}}^1)^{n_1}\;(\hat{z}^2)^{n_2}\;|0,0\rangle \qquad \text{for } n_1, n_2 \in \mathbb{N}. \tag{4.7}$$

These two formulations are equivalent since any operator-valued function $\hat{f}(\hat{\bar{z}}^1, \hat{z}^1,\; \hat{z}^2, \hat{\bar{z}}^2)$ can be obtained from a function $f(\bar{z}^1, z^1,\; z^2, \bar{z}^2)$ in the star-product formulation via the Moyal-Weyl map

$$\hat{f}(\hat{\bar{z}}^1, \hat{z}^1,\; \hat{z}^2, \hat{\bar{z}}^2) = -\frac{1}{(2\pi)^4}\int \mathrm{d}k^1\mathrm{d}\bar{k}^1\mathrm{d}z^1\mathrm{d}\bar{z}^1\; \mathrm{d}k^2\mathrm{d}\bar{k}^2\mathrm{d}z^2\mathrm{d}\bar{z}^2\; f(\bar{z}^1, z^1,\; z^2, \bar{z}^2)$$
$$\mathrm{e}^{-\mathrm{i}\bar{k}^1(\hat{z}^1 - z^1) - \mathrm{i}k^1(\hat{\bar{z}}^1 - \bar{z}^1)}\; \mathrm{e}^{-\mathrm{i}\bar{k}^2(\hat{z}^2 - z^2) - \mathrm{i}k^2(\hat{\bar{z}}^2 - \bar{z}^2)} \tag{4.8}$$

and vice versa

$$f(\bar{z}^1, z^1,\; z^2, \bar{z}^2) = 2\pi^2\theta^{12}\theta^{34}\frac{1}{(2\pi)^4}\int \mathrm{d}k^1\mathrm{d}\bar{k}^1\; \mathrm{d}k^2\mathrm{d}\bar{k}^2\; \mathrm{tr}\big(\hat{f}(\hat{\bar{z}}^1, \hat{z}^1,\; \hat{z}^2, \hat{\bar{z}}^2)$$
$$\mathrm{e}^{-\mathrm{i}\bar{k}^1(\hat{z}^1 - z^1) - \mathrm{i}k^1(\hat{\bar{z}}^1 - \bar{z}^1)}\; \mathrm{e}^{-\mathrm{i}\bar{k}^2(\hat{z}^2 - z^2) - \mathrm{i}k^2(\hat{\bar{z}}^2 - \bar{z}^2)}\big). \tag{4.9}$$

Under this map we have

$$f \star g \mapsto \hat{f}\;\hat{g} \qquad \text{and} \qquad \int_{\mathbb{R}^4} \mathrm{d}^4x\; f \mapsto 2\pi^2\;\theta^{12}\;\theta^{34}\;\mathrm{tr}(\hat{f}) \tag{4.10}$$

and the spatial derivatives become commutators

$$\begin{aligned} \partial_{z^1}\; f &\mapsto -\frac{1}{2\theta^{12}}\;[\hat{\bar{z}}^1, \hat{f}] \qquad & \partial_{\bar{z}^1}\; f &\mapsto \frac{1}{2\theta^{12}}\;[\hat{z}^1, \hat{f}] \\ \partial_{z^2}\; f &\mapsto \frac{1}{2\theta^{34}}\;[\hat{\bar{z}}^2, \hat{f}] \qquad & \partial_{\bar{z}^2}\; f &\mapsto -\frac{1}{2\theta^{34}}\;[\hat{z}^2, \hat{f}]. \end{aligned} \tag{4.11}$$

In practice (4.8) is given by the following recipe: replace coordinates by operators using the symmetric Weyl ordering.

From now on we will restrict ourselves to the case of self-dual noncommutative Euclidean space $\mathbb{R}^4_\theta$ with the non-commutativity tensor given by

$$\theta^{\mu\nu} \;=\; \theta\,\eta^{3\mu\nu}, \quad \text{i.e.} \quad \theta^{12} = \theta^{34} = \theta. \tag{4.12}$$

Let us gather some general results, which will be needed in the following calculations. First of all we have

$$\hat{r}^2|n_1,n_2\rangle = \hat{x}^\mu\hat{x}_\mu|n_1,n_2\rangle = (\hat{\bar{z}}^1\ \hat{z}^1 + \hat{z}^2\ \hat{\bar{z}}^2 + 2\theta)|n_1,n_2\rangle = 2\theta(n_1+n_2+1)|n_1,n_2\rangle. \tag{4.13}$$

Thus, every operator-valued function $\hat{f}(\hat{r}^2)$ can be expressed in terms of the basis (4.7) by

$$\hat{f}(\hat{r}^2) = \sum_{n_1,n_2\geq 0} f(2\theta(n_1+n_2+1))|n_1,n_2\rangle\langle n_1,n_2|. \tag{4.14}$$

Later on we will be interested in expressions like $\hat{x}_i^\mu = \hat{x}^\mu - b_i^\mu$. Therefore, we need the concept of coherent states [36]. Coherent states are generated by the unitary shift operator

$$D(w^1,w^2) = \mathrm{e}^{w^1\hat{\bar{z}}^1 - \bar{w}^1\hat{z}^1 + w^2\hat{\bar{z}}^2 - \bar{w}^2\hat{z}^2}, \tag{4.15}$$

where $w^1 = b^1 + \mathrm{i}b^2$ and $w^2 = b^3 - \mathrm{i}b^4$. Then one obtains the shifted operators by

$$\begin{pmatrix} \hat{z}^1 - w^1 \\ \hat{\bar{z}}^1 - \bar{w}^1 \\ \hat{z}^2 - w^2 \\ \hat{\bar{z}}^2 - \bar{w}^2 \end{pmatrix} = D(w^1,w^2) \begin{pmatrix} \hat{z}^1 \\ \hat{\bar{z}}^1 \\ \hat{z}^2 \\ \hat{\bar{z}}^2 \end{pmatrix} D^\dagger(w^1,w^2). \tag{4.16}$$

Note that every operator-valued function $\hat{f}(\hat{r}_1^2)$ can be expressed in terms of the new basis

$$\hat{f}(\hat{r}_i^2) = \sum_{n_1,n_2\leq 0} f(2\theta(n_1+n_2+1))D(b_i^\mu)\ |n_1,n_2\rangle\langle n_1,n_2|\ D^\dagger(b_i^\mu). \tag{4.17}$$

There is also a calculation rule, which we will need later on. Let $\hat{f}(\hat{r}^2)$ be any operator-valued function of $\hat{r}^2$, then

$$\begin{aligned} \hat{z}^1\ \hat{f}(\hat{r}^2) &= \hat{f}(\hat{r}^2+2\theta)\ \hat{z}^1 & \qquad \hat{\bar{z}}^1\ \hat{f}(\hat{r}^2) &= \hat{f}(\hat{r}^2-2\theta)\ \hat{\bar{z}}^1 \\ \hat{\bar{z}}^2\ \hat{f}(\hat{r}^2) &= \hat{f}(\hat{r}^2+2\theta)\ \hat{\bar{z}}^2 & \qquad \hat{z}^2\ \hat{f}(\hat{r}^2) &= \hat{f}(\hat{r}^2-2\theta)\ \hat{z}^2. \end{aligned} \tag{4.18}$$

Using (4.12), (3.20) and (3.21) as well as some relations calculated in [39], we find

$$\hat{\mathbf{x}}_i\ \hat{\mathbf{x}}_i^\dagger = \hat{r}_i^2\ \mathbf{1}_2 - 2\theta\ \sigma_3 \quad \text{and} \quad \hat{\mathbf{x}}_i^\dagger\ \hat{\mathbf{x}}_i = \hat{r}_i^2\ \mathbf{1}_2, \tag{4.19}$$

where $\hat{\mathbf{x}}_i = (\hat{x}^\mu - b_i^\mu)\ e_\mu^\dagger$ as usual. With the help of (4.18), one obtains

$$\begin{aligned} \hat{\mathbf{x}}_i\ \hat{f}(\hat{r}_i^2) &= \begin{pmatrix} \hat{f}(\hat{r}_i^2-2\theta) & 0 \\ 0 & \hat{f}(\hat{r}_i^2+2\theta) \end{pmatrix} \hat{\mathbf{x}}_i \qquad \text{and} \\ \hat{f}(\hat{r}_i^2)\ \hat{\mathbf{x}}_i^\dagger &= \hat{\mathbf{x}}_i^\dagger \begin{pmatrix} \hat{f}(\hat{r}_i^2-2\theta) & 0 \\ 0 & \hat{f}(\hat{r}_i^2+2\theta) \end{pmatrix}, \end{aligned} \tag{4.20}$$

which we will need in the following discussions. From now on - for the sake of clarity - we will omit the hats over the operators until the end of this chapter.

4.3 Noncommutative ADHM and the Incompleteness Problem

The modified ADHM construction on $\mathbb{R}^4_\theta$ can be directly taken from section 3.4 [35, 34, 21, 29, 9], since we have chosen self-dual noncommutative Euclidean space (4.12). But we need to replace the coordinates by operators. Let us take the same data Δ (3.45) as in the commutative case (see section 3.5) for the construction of an n-instanton solutions in $U(2)$ gauge theory. Then we end up with the same equations (3.51) - (3.52) but now for noncommutative coordinates. Here we additionally need to show that the noncommutative version of the completeness relation (3.41) holds.

The 't Hooft ansatz (3.53) solves (3.51) and (3.52) also in the noncommutative case [31]

$$\Psi_0^{\text{'t Hooft}} = \phi_n^{-\frac{1}{2}}\ \mathbf{1}_2 \quad \text{and} \quad \Psi_i^{\text{'t Hooft}} = -\mathbf{x}_i\ \frac{\Lambda_i}{r_i^2}\ \phi_n^{-\frac{1}{2}}, \tag{4.21}$$

with $\phi_n = 1 + \sum_{i=1}^n \frac{\Lambda_i^2}{r_i^2}$ as in (3.25), but where the r_i^2 are operators. Unfortunately, inserting this in the left hand side of (3.41), we obtain

$$\mathbf{\Psi}^{\text{'t Hooft}}\ (\mathbf{\Psi}^{\text{'t Hooft}})^\dagger + \Delta\ (\Delta^\dagger\Delta)^{-1}\Delta^\dagger = \mathbf{1}_{2n+2} - \Pi, \tag{4.22}$$

where Π is a projector,

$$\Pi = \begin{pmatrix} \mathbf{0}_2 & \dots & \mathbf{0}_2 \\ \vdots & \Pi_K & \\ \mathbf{0}_2 & & \end{pmatrix} \qquad \text{and} \qquad \Pi_K = \begin{pmatrix} \mathbf{1}_2 - x_1\frac{1}{r_1^2}x_1^\dagger & & \\ & \ddots & \\ & & \mathbf{1}_2 - x_n\frac{1}{r_n^2}x_n^\dagger \end{pmatrix}. \tag{4.23}$$

This is due to the fact that $\mathbf{x}_i^\dagger$ has zero modes $\Gamma_i = \begin{pmatrix} |b_i\rangle \\ 0 \end{pmatrix}$, where $|b_i\rangle = D(b_i^\mu)|0,0\rangle$ are coherent states introduced in section 4.2. One can see this by calculating

$$\mathbf{x}_i^\dagger\ \Gamma_i = -\mathrm{i}\ \begin{pmatrix} \bar{z}_i^2 & \bar{z}_i^1 \\ z_i^1 & -z_i^2 \end{pmatrix}\ \begin{pmatrix} |b_i\rangle \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.24}$$

We find

$$\Gamma_i^\dagger\ \Gamma_i = \mathbf{1}_2 \qquad \text{and} \qquad \Gamma_i\ \Gamma_i^\dagger = \begin{pmatrix} |b_i\rangle\langle b_i| & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.25}$$

With (4.20), (4.24) and (4.25) we can calculate

$$\mathbf{x}_i\frac{1}{r_i^2}\mathbf{x}_i^\dagger = (\mathbf{1}_2 - \Gamma_i\ \Gamma_i^\dagger)\ \mathbf{x}_i\frac{1}{r_i^2}\mathbf{x}_i^\dagger\ (\mathbf{1}_2 - \Gamma_i\ \Gamma_i^\dagger) = \mathbf{1}_2 - \Gamma_i\ \Gamma_i^\dagger. \tag{4.26}$$

In (4.26) we have projected onto a space, where the zero modes of $\mathbf{x}_i^\dagger$ are removed.

Thus, we find that for (4.21), the completeness relation (3.41) in the noncommutative case is not satisfied due to (4.22). Nevertheless, we calculate the field strength

$$F_{\mu\nu} = 2\ \Psi^\dagger D\,(\Delta^\dagger\Delta)^{-1}\eta_{\mu\nu}\ D^\dagger\Psi + \phi_n^{-\frac{1}{2}}\begin{pmatrix} \sum_{i=1}^n \frac{\Lambda_i^2}{4\theta^2}\ |b_i\rangle\langle b_i| & 0 \\ 0 & 0 \end{pmatrix}\phi_n^{-\frac{1}{2}}\ 2\ \bar{\eta}_{\mu\nu} \tag{4.27}$$

analogously to (3.44). Now we can see directly that in (4.27) both, the self-dual and the anti-self-dual part, are non-zero.

4.4 ’t Hooft Solution in Noncommutative Coordinates

In the following, we will show how to tackle the incompleteness of the ’t Hooft solution from section 4.3. One can transform $\Psi_0^{\text{'t Hooft}}$ and $\Psi_i^{\text{'t Hooft}}$ in a way such that the completeness relation (3.41) also holds in noncommutative coordinates [52]. We will use a suitable notation in order to have a different view on the incompleteness problem (see section 4.3). This notation will also guarantee an easy access to a generalization to some arbitrary ADHM data Δ. Therefore let us introduce the following decomposition of Δ:

$$\Delta = \begin{pmatrix} K \\ \Delta_K \end{pmatrix}. \tag{4.28}$$

Then we demand as in (3.36) that

$$\Delta^\dagger\ \Delta = K^\dagger\ K + \Delta_K^\dagger\ \Delta_K = \mathbf{1}_2 \otimes f^{-1} = \mathbf{f}^{-1} \tag{4.29}$$

which will guarantee that (3.34) and (3.35) are fulfilled. Furthermore, we will gather the Ψ_i for $i = 1, \ldots, n$ in one vector

$$\Psi_K = \begin{pmatrix} \Psi_1 \\ \vdots \\ \Psi_n \end{pmatrix}. \tag{4.30}$$

Then the completeness relation (3.41) in noncommutative coordinates reads

$$\begin{pmatrix} \Psi_0\ \Psi_0^\dagger & \Psi_0\ \Psi_K^\dagger \\ \Psi_K\ \Psi_0^\dagger & \Psi_K \Psi_K^\dagger \end{pmatrix} = \begin{pmatrix} \mathbf{1}_2 - K\ \mathbf{f}\ K^\dagger & -K\ \mathbf{f}\ \Delta_K^\dagger \\ -\Delta_K\ \mathbf{f}\ K^\dagger & \mathbf{1}_{2n} - \Delta_K\ \mathbf{f}\ \Delta_K^\dagger \end{pmatrix}. \tag{4.31}$$

Let us now consider the ansatz

$$\begin{aligned} &\Psi_0 = (\mathbf{1}_2 - K\ \mathbf{f}\ K^\dagger)^{\frac{1}{2}} \qquad \text{and} \\ &\Psi_K = -\Delta_K\ \mathbf{f}\ K^\dagger (\mathbf{1}_2 - K\ \mathbf{f}\ K^\dagger)^{-\frac{1}{2}}. \end{aligned} \tag{4.32}$$

It is easy to check that this choice (4.32) satisfies the noncommutative version of (3.37) and (3.38). Furthermore, it solves the identity of (4.31) by construction for the (11)-, the (12)- and the (21)-elements. But the (22)-elements on both sides in (4.31) are not necessarily equal, which we will see in the following. With the help of (4.29), one can calculate

$$\begin{aligned} &\Delta_K^\dagger\ (\mathbf{1}_{2n} - \Delta_K\ \mathbf{f}\ \Delta_K^\dagger - \Psi_K\ \Psi_K^\dagger) \\ &= (\mathbf{1}_{2n} - \Delta_K^\dagger\ \Delta_K\ \mathbf{f})\ \Delta_K^\dagger - \Delta_K^\dagger\ \Delta_K\ \mathbf{f}\ K^\dagger (\mathbf{1}_2 - K\ \mathbf{f}\ K^\dagger)^{-1}\ K\ \mathbf{f}\ \Delta_K^\dagger \\ &= \big(\mathbf{1}_{2n} - (\mathbf{f}^{-1} - K^\dagger\ K)\ \mathbf{f}\big)\ \Delta_K^\dagger \\ &- (\mathbf{f}^{-1} - K^\dagger\ K)\ \mathbf{f}\ K^\dagger (\mathbf{1}_2 - K\ \mathbf{f}\ K^\dagger)^{-1}\ K\ \mathbf{f}\ \Delta_K^\dagger \\ &= K^\dagger\ K\ \mathbf{f}\ \Delta_K^\dagger - K^\dagger\ K\ \mathbf{f}\ \Delta_K^\dagger = 0. \end{aligned} \tag{4.33}$$

But the bracket in the first line of (4.33) is not necessarily equal to zero, which is due to the fact that $\Delta_K^\dagger$ may have zero modes.

To become more concrete, we consider the noncommutative version of the ADHM data (3.45) from section 3.5. With the decomposition (4.28) it becomes

$$K = (\Lambda_1\ \mathbf{1}_2 \ldots \Lambda_n\ \mathbf{1}_2) = \mathbf{1}_2 \otimes \Lambda^T \qquad \text{and} \qquad \Delta_K = \begin{pmatrix} \mathbf{x}_1 & & \mathbf{0}_2 \\ & \ddots & \\ \mathbf{0}_2 & & \mathbf{x}_n \end{pmatrix}. \tag{4.34}$$

We find

$$K^\dagger\ K + \Delta_K^\dagger\ \Delta_K = \mathbf{1}_2 \otimes (R + \Lambda\ \Lambda^T) = \mathbf{1}_2 \otimes (R^{-1} - R^{-1}\ \Lambda\ \phi_n^{-1}\ \Lambda^T\ R^{-1})^{-1}, \tag{4.35}$$

where Λ, R (cf. (3.47)) and $\phi_n = 1 + \Lambda^T R^{-1}\Lambda$ (cf. (3.49)) are defined exactly as in section 3.5 but with r_i^2 now being operators. Note that the ansatz (4.32) for Ψ_0 and Ψ_K coincides with the 't Hooft ansatz $\mathbf{\Psi}^{\text{'t Hooft}}$ (4.21). By direct calculations we find

$$\mathbf{1}_2 - K\ \mathbf{f}\ K^\dagger = \mathbf{1}_2 \otimes (1 - \Lambda^T R^{-1}\Lambda + \Lambda^T R^{-1}\Lambda\ \phi_n^{-1}\ \Lambda^T R^{-1}\Lambda) = \begin{pmatrix} \phi_n^{-1} & 0 \\ 0 & \phi_n^{-1} \end{pmatrix}. \tag{4.36}$$

Inserting (4.36) into (4.32) we obtain

$$\begin{aligned} &\Psi_0 = \mathbf{1}_2 \otimes \phi_n^{-\frac{1}{2}} = \Psi_0^{\text{'t Hooft}} \qquad \text{and} \\ &\Psi_K = -\Delta_K\ (\mathbf{1}_2 \otimes R^{-1}\ \Lambda\ \phi_n^{-\frac{1}{2}}) = \Psi_K^{\text{'t Hooft}}. \end{aligned} \tag{4.37}$$

We already know from (4.22) that

$$\mathbf{\Psi}_K^{\text{'t Hooft}}\ (\mathbf{\Psi}_K^{\text{'t Hooft}})^\dagger = \mathbf{1}_{2n} - \Delta_K\ \mathbf{f}\ \Delta_K^\dagger - \Pi_K, \tag{4.38}$$

where (using (4.23) and (4.26))

$$\Pi_K = \begin{pmatrix} \Gamma_1\ \Gamma_1^\dagger & & \\ & \ddots & \\ & & \Gamma_n\ \Gamma_n^\dagger \end{pmatrix}$$

with Γ_i being the zero modes of $\mathbf{x}_i^\dagger$. This is why the completeness relation (3.41) is not satisfied. We have already obtained this result in (4.33) for any ADHM data, when $\Delta_K^\dagger$ has zero modes.

For some arbitrary ADHM data suppose that generally $\mathbf{\Gamma}_n$ is a $2n \times n$ matrix consisting of all zero modes of $\Delta_K^\dagger$ and $\mathbf{\Gamma}_n^\dagger\ \mathbf{\Gamma}_n = \mathbf{1}_n$. One should try to remove this zero modes from $\mathbf{1}_{2n} - \Delta_K\ \mathbf{f}\ \Delta_K^\dagger - \Psi_K\ \Psi_K^\dagger$. Therefore, let us introduce the 2×2 matrix shift operator S_n, which satisfies

$$S_n\ S_n^\dagger = \mathbf{1}_2 \quad \text{and} \quad S_n^\dagger\ S_n = \begin{pmatrix} 1 - \sum_{k=0}^{n-1} |k,0\rangle\langle k,0| & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.39}$$

as well as the $n \times 2$ matrix

$$Y_n = \begin{pmatrix} \langle 0,0| & 0 \\ \vdots & \vdots \\ \langle n-1,0| & 0 \end{pmatrix}, \tag{4.40}$$

which satisfies

$$Y_n\ Y_n^\dagger = \mathbf{1}_n \qquad \text{and} \qquad Y_n^\dagger\ Y_n = \mathbf{1}_2 - S_n^\dagger\ S_n. \tag{4.41}$$

Then one finds that

$$\begin{aligned} &\tilde{\Psi}_0 = \Psi_0\ S_n = (\mathbf{1}_2 - K\ \mathbf{f}\ K^\dagger)^{\frac{1}{2}}\ S_n \quad \text{and} \\ &\tilde{\Psi}_K = \mathbf{\Gamma}_n\ Y_n + \Psi_K\ S_n = \mathbf{\Gamma}_n\ Y_n - \Delta_K\ \mathbf{f}\ K^\dagger\ (\mathbf{1}_2 - K\ \mathbf{f}\ K^\dagger)^{-\frac{1}{2}}\ S_n \end{aligned} \tag{4.42}$$

solves the noncommutative version of the completeness relation (4.31). Furthermore, the ansatz (4.42) still solves (3.37) and (3.38). By direct calculation we find, using (4.29),

$$K^\dagger\tilde{\Psi}_0 + \Delta_K^\dagger\tilde{\Psi}_K = \left(K^\dagger(\mathbf{1}_2 - K\mathbf{f}K^\dagger)^{\frac{1}{2}} - (\mathbf{f}^{-1} - K^\dagger K)\mathbf{f}K^\dagger(\mathbf{1}_2 - K\mathbf{f}K^\dagger)^{-\frac{1}{2}}\right) S_n = 0 \quad (4.43)$$

and with (4.39) and (4.41) we obtain

$$\begin{aligned}
&\tilde{\Psi}_0^\dagger\ \tilde{\Psi}_0 + \tilde{\Psi}_K^\dagger\ \Psi_K \\
&= S_n^\dagger\ (\mathbf{1}_2 - K\ \mathbf{f}\ K^\dagger)\ S_n + Y_n^\dagger\ \mathbf{\Gamma}_n^\dagger\ \mathbf{\Gamma}_n\ Y_n \\
&+ S_n^\dagger\ (\mathbf{1}_2 - K\ \mathbf{f}\ K^\dagger)^{-\frac{1}{2}}\ K\ \mathbf{f}\ \Delta_K^\dagger\ \Delta_K\ \mathbf{f}\ K^\dagger\ (\mathbf{1}_2 - K\ \mathbf{f}\ K^\dagger)^{-\frac{1}{2}}\ S_n \\
&= S_n^\dagger\ (\mathbf{1}_2 - K\ \mathbf{f}\ K^\dagger)^{\frac{1}{2}}\ (\mathbf{1}_2 - K\ \mathbf{f}\ K^\dagger + K\ \mathbf{f}\ K^\dagger)\ (\mathbf{1}_2 - K\ \mathbf{f}\ K^\dagger)^{-\frac{1}{2}} S_n + Y_n^\dagger\ \mathbf{\Gamma}_n^\dagger\ \mathbf{\Gamma}_n\ Y_n \\
&= S_n^\dagger\ S_n + Y_n^\dagger\ Y_n = \mathbf{1}_2.
\end{aligned} \quad (4.44)$$

Note that with (4.39) we have $\tilde{\Psi}_0\ \tilde{\Psi}_0^\dagger = \Psi_0\ \Psi_0^\dagger$ and

$$\tilde{\Psi}_K\ \tilde{\Psi}_0^\dagger = \mathbf{\Gamma}_n\ Y_n\ S_n^\dagger\ \Psi_0^\dagger + \Psi_K\ S_n\ S_n^\dagger\ \Psi_0^\dagger = \Psi_K\ \Psi_0^\dagger. \quad (4.45)$$

In (4.45) we have additionally used the fact that

$$(Y_n\ S_n^\dagger)^\dagger = S_n\ S_n^\dagger\ (Y_n\ S_n^\dagger)^\dagger = S_n\ (Y_n\ S_n^\dagger\ S_n)^\dagger = 0. \quad (4.46)$$

We come back to the special ADHM data (4.34). Here $\mathbf{\Gamma}_n$ simply is given by

$$\mathbf{\Gamma}_n = \begin{pmatrix} |b_1\rangle & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & & |b_n\rangle \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad (4.47)$$

which satisfies

$$\Delta_K^\dagger\ \mathbf{\Gamma}_n = 0, \qquad \mathbf{\Gamma}_n^\dagger\ \mathbf{\Gamma}_n = \mathbf{1}_n \qquad \text{and} \qquad \mathbf{\Gamma}_n\ \mathbf{\Gamma}_n^\dagger = \Pi_K \quad (4.48)$$

with Π_K as in (4.23). Using (4.38), (4.48), (4.39), (4.41) and (4.46), we obtain

$$\tilde{\Psi}_K^{\text{'t Hooft}}\ (\tilde{\Psi}_K^{\text{'t Hooft}})^\dagger = \mathbf{\Gamma}_n\ Y_n\ Y_n^\dagger\ \mathbf{\Gamma}_n^\dagger + \mathbf{1}_{2n} - \Delta_K\ \mathbf{f}\ \Delta_K^\dagger - \Pi_K = \mathbf{1}_{2n} - \Delta_K\ \mathbf{f}\ \Delta_K^\dagger. \quad (4.49)$$

Thus, we have found directly that equation (4.31) is also solved for the (22)-elements. Hence, we have found a noncommutative generalization of the 't Hooft instantons, i.e. a n-instanton solution in a singular gauge.

Let us make the following remark: as we have seen above, the ansatz (4.42) solves the noncommutative version of the completeness relation (4.31) as well as (3.37) and (3.38) generally for any ADHM data Δ satisfying (3.34) and (3.35), when $\mathbf{\Gamma}_n$ in (4.42) is a matrix consisting of all zero modes of $\Delta_K^\dagger$. This is still valid for the general Euclidean space $\mathbb{R}_\theta^4$ with $\theta^{34} < 0$. For the Euclidean space $\mathbb{R}_\theta^4$ with $\theta^{34} > 0$, one needs another concept [52] since in this case Δ_K may have zero modes. Furthermore, one can easily generalize (4.42) to the case with the gauge group $U(N)$.

We have already mentioned that the n instanton solutions obtained by $\tilde{\mathbf{\Psi}}^{\text{'t Hooft}}$ are given in a singular gauge and need to be transformed by some noncommutative version of the singular gauge transformations from section 3.3 which are the so-called Murray-von-Neumann transformations [31].

For one instanton, the explicit form of $g_1 \in U(2)$, generating the transformation $\mathbf{\Psi}^{\text{regular}} = \tilde{\mathbf{\Psi}}^{\text{'t Hooft}}\ g_1$, is given by

$$g_1 = S_1^\dagger\ \frac{1}{\sqrt{r_1^2}}\ \mathbf{x}_1^\dagger - \Lambda_1\ Y_1^\dagger\ \mathbf{\Gamma}_1^\dagger\ f_1 \tag{4.50}$$

with

$$f_1 = \begin{pmatrix} \frac{1}{\sqrt{r_1^2 - 2\theta + \Lambda_1^2}} & 0 \\ 0 & \frac{1}{\sqrt{r_1^2 + 2\theta + \Lambda_1^2}} \end{pmatrix}. \tag{4.51}$$

We check that $g_1 \in U(2)$ by calculating

$$\begin{aligned} g_1^\dagger\ g_1 &= \mathbf{x}_1\ \frac{1}{r_1^2}\ \mathbf{x}_1^\dagger + \Lambda_1^2\ f_1^\dagger\ \mathbf{\Gamma}_1\ Y_1\ Y_1^\dagger\ \mathbf{\Gamma}_1^\dagger f_1 = \mathbf{1}_2 \qquad \text{and} \\ g_1\ g_1^\dagger &= S_1^\dagger\ S_1 + \Lambda_1^2\ Y_1^\dagger\ \mathbf{\Gamma}_1^\dagger f_1\ f_1^\dagger\ \mathbf{\Gamma}_1\ Y_1 = \mathbf{1}_2 \end{aligned} \tag{4.52}$$

We will discuss $\mathbf{\Psi}^{\text{regular}}$ later on in section 4.6.

For two instantons g_2 is given by

$$g_2 = S_2^\dagger\ \phi_2^{\frac{1}{2}}\ \mathbf{x}_1^\dagger\ (\mathbf{b}_2 - \mathbf{b}_1)\ \mathbf{x}_2^\dagger\ f_2 - Y_2^\dagger\ \mathbf{\Gamma}_2^\dagger\ \begin{pmatrix} \Lambda_1\ (\mathbf{b}_2 - \mathbf{b}_1)\ \mathbf{x}_2^\dagger \\ \Lambda_2\ (\mathbf{b}_2 - \mathbf{b}_1)\ \mathbf{x}_1^\dagger \end{pmatrix} f_2. \tag{4.53}$$

Demanding that $g_2 \in U(2)$ leads to equations for f_2 which will again appear in section 4.6. Note that then $\mathbf{\Psi}^{\text{regular}} = \tilde{\mathbf{\Psi}}^{\text{'t Hooft}}\ g_2$ will also fulfill the completeness relation.

4.5 Multi-Instantons with Noncommutative Translational Moduli

There is another possibility to overcome the problem with the 't Hooft ansatz from section 4.3. Moreover, this approach will be interesting in its own right. Namely, let us consider the case where the translational moduli are noncommutative as well [26]. That means that we now have the following commutation relations:

$$[x^\mu, x^\nu] = \mathrm{i}\theta^{\mu\nu}\ , \qquad [b_i^\mu, b_j^\nu] = -\mathrm{i}\theta^{\mu\nu}\delta_{ij} \qquad \text{and} \qquad [x^\mu, b_i^\nu] = 0. \tag{4.54}$$

Here our choice (4.54) is of course very specific. Then we find

$$[x_i^\mu, x_j^\nu] = \mathrm{i}\theta^{\mu\nu}\,(1 - \delta_{ij}) \qquad \text{for} \quad i, j = 1, \ldots, n \tag{4.55}$$

and obtain in distinction to (4.19)

$$\mathbf{x}_i^\dagger\ \mathbf{x}_i = \mathbf{x}_i\ \mathbf{x}_i^\dagger = r_i^2\ \mathbf{1}_2. \tag{4.56}$$

For the non-commuting moduli the 't Hooft ansatz (4.21) is unchanged and solves (3.51) and (3.52). But if we compute the left hand side of (3.41), we find that the completeness relation is satisfied due to the modified commutation relations (4.54), especially due to the fact that now x_i^μ commutes with x_i^ν on the diagonal. Indeed, by direct calculations we find

$$\Delta(\Delta^\dagger\Delta)^{-1}\Delta^\dagger = \begin{pmatrix} \mathbf{1}_2 - \phi_n^{-1}\mathbf{1}_2 & \phi_n^{-1}\frac{\Lambda_1}{r_1^2}\mathbf{x}_1^\dagger & \dots & \phi_n^{-1}\frac{\Lambda_n}{r_n^2}\mathbf{x}_n^\dagger \\ \mathbf{x}_1\frac{\Lambda_1}{r_1^2}\phi_n^{-1} & \mathbf{1}_2 - \mathbf{x}_1\frac{\Lambda_1^2}{r_1^2\phi_n r_1^2}\mathbf{x}_1^\dagger & \dots & -\mathbf{x}_1\frac{\Lambda_1}{r_1^2}\phi_n^{-1}\frac{\Lambda_n}{r_n^2}\mathbf{x}_n^\dagger \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{x}_n\frac{\Lambda_n}{r_n^2}\phi_n^{-1} & -\mathbf{x}_n\frac{\Lambda_n}{r_n^2}\phi_n^{-1}\frac{\Lambda_1}{r_1^2}\mathbf{x}_1^\dagger & \dots & \mathbf{1}_2 - \mathbf{x}_n\frac{\Lambda_n^2}{r_n^2\phi_n r_n^2}\mathbf{x}_n^\dagger \end{pmatrix}$$

and

$$\mathbf{\Psi}\ \mathbf{\Psi}^\dagger = \begin{pmatrix} \phi_n^{-1}\mathbf{1}_2 & -\phi_n^{-1}\frac{\Lambda_1}{r_1^2}\mathbf{x}_1^\dagger & \dots & -\phi_n^{-1}\frac{\Lambda_n}{r_n^2}\mathbf{x}_n^\dagger \\ -\mathbf{x}_1\frac{\Lambda_1}{r_1^2}\phi_n^{-1} & \mathbf{x}_1\frac{\Lambda_1^2}{r_1^2\phi_n r_1^2}\mathbf{x}_1^\dagger & \dots & \mathbf{x}_1\frac{\Lambda_1}{r_1^2}\phi_n^{-1}\frac{\Lambda_n}{r_n^2}\mathbf{x}_n^\dagger \\ \vdots & \vdots & \ddots & \vdots \\ -\mathbf{x}_n\frac{\Lambda_n}{r_n^2}\phi_n^{-1} & \mathbf{x}_n\frac{\Lambda_n}{r_n^2}\phi_n^{-1}\frac{\Lambda_1}{r_1^2}\mathbf{x}_1^\dagger & \dots & \mathbf{x}_n\frac{\Lambda_n^2}{r_n^2\phi_n r_n^2}\mathbf{x}_n^\dagger \end{pmatrix}.$$

Note that one should be concerned about the invertibility of r_i^2. The commutation relations of the translational moduli (4.54) read similar to (4.6),

$$[w_i^1, \bar{w}_i^1] = -2\theta \qquad \text{and} \qquad [w_i^2, \bar{w}_i^2] = 2\theta, \tag{4.57}$$

where $w_i^1 = b_i^1 + \mathrm{i}b_i^2$ and $w_i^2 = b_i^3 - \mathrm{i}b_i^4$ as in section 4.2. This can be realized by now regarding also $\bar{w}_i^1,\ w_i^2$ as annihilation operators $\hat{w}_i^1,\ \hat{w}_i^2$ and $w_i^1,\ \bar{w}_i^2$ as creation operators $\hat{\bar{w}}_i^1, \hat{\bar{w}}_i^2$ which act together with $\hat{\bar{z}}^1,\ \hat{z}^2$ and $\hat{z}^1, \hat{\bar{z}}^2$ on the tensor product $\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n$ of $n{+}1$ copies of the two-oscillator Fock space. In this formulation one finds that

$$r_i^2 = (\hat{z}^1 - \hat{w}_i^1)\ (\hat{\bar{z}}^1 - \hat{\bar{w}}_i^1) + (\hat{\bar{z}}^2 - \hat{\bar{w}}_i^2)\ (\hat{z}^2 - \hat{w}_i^2) = \hat{z}_i^1\ \hat{\bar{z}}_i^1 + \hat{\bar{z}}_i^2\ \hat{z}_i^2 \tag{4.58}$$

with new creation operators

$$\hat{\bar{z}}_i^1 = \hat{\bar{z}}^1 - \hat{\bar{w}}_i^1 \qquad \text{and} \qquad \hat{z}_i^2 = \hat{z}^2 - \hat{w}_i^1 \tag{4.59}$$

and new annihilation operators

$$\hat{z}_i^1 = \hat{z}^1 - \hat{w}_i^1 \qquad \text{and} \qquad \hat{\bar{z}}_i^2 = \hat{\bar{z}}^2 - \hat{\bar{w}}_i^1 \tag{4.60}$$

created by a Bogoliubov transformation [36]. The general form of such a transformation for two sets of creation and annihilation operators $\{\alpha_0, \alpha_0^\dagger\}$ and $\{\alpha_1, \alpha_1^\dagger\}$ is

$$\begin{pmatrix} \tilde{\alpha}_0 \\ \tilde{\alpha}_1 \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} + \begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{pmatrix} \begin{pmatrix} \alpha_0^\dagger \\ \alpha_1^\dagger \end{pmatrix} \tag{4.61}$$

where $(c_{ij}) = C$ is a symmetric 2×2 matrix with complex entries. Note that the new annihilation operators $\tilde{\alpha}_0$ and $\tilde{\alpha}_1$ have normalizable vectors in their kernel only if the hermitian matrix $\mathbf{1}_2 - CC^\dagger$ is positive definite. The case under consideration is degenerate because

$$C = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \qquad \Rightarrow \qquad \mathbf{1}_2 - CC^\dagger = \mathbf{0}_2. \tag{4.62}$$

Hence, in our case $\hat{z}_i^1$ and $\hat{\bar{z}}_i^2$ as well as their hermitian conjugates have no normalizable zero modes either on $\mathcal{H}$ or on its dual $\mathcal{H}^*$. We conclude that for any $i = 1, \ldots, n$, the operator r_i^2 is invertible on all finite-norm states, i.e. on $\mathcal{H}$ as well as on $\mathcal{H}^*$.

In a number of papers it was argued that on $\mathbb{R}^4_\theta$ the topological charge of noncommutative ADHM n-instantons for the gauge group $U(N)$ is an integer since this is encoded in the dimensions of the ADHM matrices even in the noncommutative case. Noncommutativity of translational moduli does not alter this conclusion. However, due to the noncommutativity of not only x^μ but also b_i^μ, our operators are defined on the larger Fock space $\mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n$. That is why the topological charge of our solution will be n times the identity operator in $\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n$. In the commutative limit, this identity operator becomes unity, and we recover the standard 't Hooft multi-instanton with commutative moduli featuring a topological charge equal to n.

Note that this configuration with noncommutative translational moduli can be interpreted as a four-dimensional "slice" of a solution to the generalized self-dual Yang-Mills equations [53, 12, 27] on $\mathbb{R}^{4+4n}$ in which both x^μ and b_i^μ play the role of noncommutative coordinates.

4.6 Nonsingular Instantons in Noncommutative Coordinates

In this section our aim is to generalize the construction of section 3.5 to the noncommutative case for the gauge group $G = U(2)$.

One instanton. This was already calculated by Furuuchi [18]. One starts as in the commutative case (3.54) with

$$\Psi_0^{\text{regular}} = \mathbf{x}_1^\dagger \ f_1 \qquad \text{and} \qquad \Psi_1^{\text{regular}} = -\Lambda_1 \ f_1, \tag{4.63}$$

which solves (3.51). From (3.52) we obtain with the help of (4.19) the normalization matrix, which is given by

$$f_1 = \begin{pmatrix} \frac{1}{\sqrt{r_1^2 - 2\theta + \Lambda_1^2}} & 0 \\ 0 & \frac{1}{\sqrt{r_1^2 + 2\theta + \Lambda_1^2}} \end{pmatrix}$$

(cf. (4.51)). This is obviously nonsingular. We have

$$\mathbf{\Psi}^{\text{regular}} \ (\mathbf{\Psi}^{\text{regular}})^\dagger = \begin{pmatrix} \mathbf{x}_1^\dagger \ f_1^2 \ \mathbf{x}_1 & -\Lambda_1 \mathbf{x}_1^\dagger \ f_1^2 \\ -\Lambda_1 \ f_1^2 \ \mathbf{x}_1 & \Lambda_1^2 \ f_1^2 \end{pmatrix}$$

and

$$\Delta \ (\Delta^\dagger \ \Delta)^{-1} \ \Delta^\dagger = \begin{pmatrix} \frac{\Lambda^2}{r_1^2 + \Lambda_1^2} \ \mathbf{1}_2 & \frac{\Lambda_1}{r_1^2+\Lambda_1^2} \mathbf{x}_1^\dagger \\ \mathbf{x}_1 \ \frac{\Lambda_1}{r_1^2+\Lambda_1^2} & \mathbf{x}_1 \ \frac{1}{r_1^2+\Lambda_1^2} \ \mathbf{x}_1^\dagger \end{pmatrix}.$$

Using (4.20), we find that the completeness relation (3.41) is satisfied. Then one can calculate together with (3.39) and (4.11) the gauge fields

$$\begin{aligned} A_{z^a} &= (-1)^a \frac{1}{2\theta} \ (\Psi_0^\dagger \ [\bar{z}^a, \Psi_0] + \Psi_1^\dagger \ [\bar{z}^a, \Psi_1]) \\ A_{\bar{z}^a} &= (-1)^{a+1} \frac{1}{2\theta} \ (\Psi_0^\dagger \ [z^a, \Psi_0] + \Psi_1^\dagger \ [z^a, \Psi_1]) \end{aligned} \tag{4.64}$$

for $a = 1, 2$. We obtain

$$A_{z^1} = \begin{pmatrix} -\frac{\bar{z}^1}{2\theta}\left(\sqrt{\frac{r_1^2-2\theta+\Lambda_1^2}{r_1^2+\Lambda_1^2}} - 1\right) & 0 \\ -\bar{z}^2 \frac{1}{\sqrt{(r_1^2-2\theta+\Lambda_1^2)\ (r_1^2+\Lambda_1^2)}} & -\frac{\bar{z}^1}{2\theta}\left(\sqrt{\frac{r_1^2+4\theta+\Lambda_1^2}{r_1^2+2\theta+\Lambda_1^2}} - 1\right) \end{pmatrix},$$

$$A_{z^2} = \begin{pmatrix} \left(\sqrt{\frac{r_1^2-2\theta+\Lambda_1^2}{r_1^2+\Lambda_1^2}} - 1\right)\frac{\bar{z}^2}{2\theta} & -\frac{1}{\sqrt{(r_1^2-2\theta+\Lambda_1^2)\ (r_1^2+\Lambda_1^2)}}\bar{z}^1 \\ 0 & \left(\sqrt{\frac{r_1^2+4\theta+\Lambda_1^2}{r_1^2+2\theta+\Lambda_1^2}} - 1\right)\frac{\bar{z}^2}{2\theta} \end{pmatrix} \tag{4.65}$$

and $A_{\bar{z}^a} = -A_{z^a}^\dagger$ for $a = 1, 2$. This coincides with the calculation done in the dressing approach [24]. In the commutative limit ($\theta \to 0$), this solution coincides with the BPST solution [6].

Two instantons. Again as in the commutative case (3.56) we consider the ansatz

$$\Psi_0^{\text{regular}} = \mathbf{x}_1^\dagger\ (\mathbf{b}_2 - \mathbf{b}_1)\ \mathbf{x}_2^\dagger\ f_2 = \mathbf{x}_2^\dagger\ (\mathbf{b}_2 - \mathbf{b}_1)\ \mathbf{x}_1^\dagger\ f_2,$$
$$\Psi_1^{\text{regular}} = -\Lambda_1\ (\mathbf{b}_2 - \mathbf{b}_1)\ \mathbf{x}_2^\dagger\ f_2 \qquad \text{and} \qquad \Psi_2^{\text{regular}} = -\Lambda_2\ (\mathbf{b}_2 - \mathbf{b}_1)\ \mathbf{x}_1^\dagger\ f_2, \tag{4.66}$$

which solves (3.51). The second equality sign in Ψ_0^{regular} is still valid in the noncommutative case. Namely, with (4.19) we have

$$\begin{aligned}\mathbf{x}_1^\dagger\ (\mathbf{b}_2 - \mathbf{b}_1)\ \mathbf{x}_2^\dagger &= \mathbf{x}_1^\dagger\ (\mathbf{x}_1 - \mathbf{x}_2)\ \mathbf{x}_2^\dagger = r_1^2\ \mathbf{x}_2^\dagger - \mathbf{x}_1^\dagger\ (r_2^2\ \mathbf{1}_2 - 2\theta\ \sigma_3) \\ &= \mathbf{x}_2^\dagger\ r_1^2 - r_2^2\ \mathbf{x}_1^\dagger - 2\theta\ \mathbf{x}_2^\dagger\ \sigma_3 = \mathbf{x}_2^\dagger\ (\mathbf{x}_1 - \mathbf{x}_2)\ \mathbf{x}_1^\dagger = \mathbf{x}_2^\dagger\ (\mathbf{b}_2 - \mathbf{b}_1)\ \mathbf{x}_1^\dagger\end{aligned}$$

since $[\mathbf{x}_i^\dagger, r_j^2\ \mathbf{1}_2] = 2\theta\ \mathbf{x}_j^\dagger\ \sigma_3$. We then calculate

$$\begin{aligned}&(\boldsymbol{\Psi}^{\text{regular}})^\dagger\ \boldsymbol{\Psi}^{\text{regular}} \\ &= f_2^\dagger\ \left(\mathbf{x}_2(\mathbf{b}_2^\dagger - \mathbf{b}_1^\dagger)\mathbf{x}_1\ \mathbf{x}_1^\dagger(\mathbf{b}_2 - \mathbf{b}_1)\mathbf{x}_2^\dagger + (b_2 - b_1)^2\left(\Lambda_1^2(r_2^2\ \mathbf{1}_2 - 2\theta\sigma_3) + \Lambda_2^2(r_1^2\ \mathbf{1}_2 - 2\theta\sigma_3)\right)\right)\ f_2 \\ &= f_2^\dagger\ \Big((b_2 - b_1)^2\left(\frac{1}{2}(r_1^2\ r_2^2 + r_2^2\ r_1^2) + \Lambda_1^2\ r_2^2 + \Lambda_2^2\ r_1^2 + 4\theta^2\right)\ \mathbf{1}_2 \\ &- 2\theta\left((b_2 - b_1)^2\left(\frac{1}{2}(r_1^2 + r_2^2) + \Lambda_1^2 + \Lambda_2^2\right) + \frac{1}{2}(r_1^2 - r_2^2)^2 - \frac{1}{8\theta^2}([r_1^2, r_2^2])^2\right)\ \sigma_3 \\ &- 2\theta\left(x_1^\mu\ x_2^\nu\ \eta_{\mu\nu}^2(r_1^2 - r_2^2) - \frac{i}{2\theta}\ x_1^\mu\ x_2^\nu\ \eta_{\mu\nu}^1[r_1^2, r_2^2]\right)\ \sigma_1 \\ &- 2\theta\left(-x_1^\mu\ x_2^\nu\ \eta_{\mu\nu}^1(r_1^2 - r_2^2) - \frac{i}{2\theta}\ x_1^\mu\ x_2^\nu\ \eta_{\mu\nu}^2[r_1^2, r_2^2]\right)\ \sigma_2\Big)\ f_2.\end{aligned}$$

This is a 2×2 matrix with non-commuting entries, which is non-diagonal. One still has to find f_2 such that $(\boldsymbol{\Psi}^{\text{regular}})^\dagger\ \boldsymbol{\Psi}^{\text{regular}} = \mathbf{1}_2$. This is a rather complicated task and we do not try to do it here. We know from section 4.4 that the ansatz (4.66) will fulfill the completeness relation. We expect this solution to be nonsingular at finite points.

Multi-instantons. The construction of section 3.5 can hopefully be generalized for many instantons. But even for the case of three instantons formulas become very complicated due to the non-commutativity. To summarize this section, for the gauge group $U(2)$ only the single-instanton solution in a nonsingular gauge at finite points is explicitly known on noncommutative $\mathbb{R}_\theta^4$.

5 Monopoles in Commutative Coordinates

5.1 Monopoles and their Applications in Physics

Magnetic monopoles in classical electrodynamics are the magnetic equivalent of point electric charges. Up to now they have not been observed experimentally. Due to this absence of magnetic charges the electric-magnetic duality of Maxwell's equations is spoiled in nature. But if we want to introduce magnetic charges it occurs that

$$\vec{B} = \vec{\nabla} \times \vec{A} \;\Rightarrow\; \vec{\nabla} \cdot \vec{B} = 0, \tag{5.1}$$

where $\vec{B}$ is the magnetic field and $\vec{A}$ the smooth vector potential on $\mathbb{R}^3$. The implication (5.1) tells us that in general we cannot have magnetic charges. In 1931, Dirac found [15] that this problem can be overcome by adding some term to the magnetic potential, which is singular on a line from the monopole to infinity, the Dirac string, and zero everywhere else. In order to describe this situation, we need two vector potentials, which are not defined everywhere but are related by a transition function (cf. chapter 2) on the overlap of two regions. Furthermore, Dirac has shown that the existence of magnetic monopoles can explain the quantization of the electric charge. For a review on magnetic monopoles see e.g. [48].

In pure Yang-Mills theory in $3 + 1$ dimensions there are no regular localized solutions [10]. One needs to extend this theory by a scalar field, the so-called Higgs field. In 1974, 't Hooft and Polyakov independently found a magnetic monopole for Yang-Mills-Higgs theory with gauge group $SO(3)$ broken down to $U(1)$ [51, 37]. They argued that monopoles exist in any unified gauge theory, which has $U(1)$ as a subgroup. Moreover, here monopoles are unavoidable. Note that these monopoles may not have a Dirac string.

In QCD models, monopoles can help to explain confinement. Mandelstam and 't Hooft suggested that the vacuum consist of monopoles [32, 50]. If these monopoles condense, it will give a magnetically superconducting ground state, which confines quarks as follows. The color electric field of the heavy quarks would have a tube-like structure in the vacuum state of gluodynamics such that the quarks cannot penetrate the superconductor. This is called the superconductor model of confinement.

Another important application is the monopole catalysis of the proton decay [43]. Grand unified theories generally predict a proton decay but if it is considered in the background of monopoles, then the cross sections will be much larger. The reactions are of the following type:

$$\text{proton} + \text{monopole} \rightarrow \text{electron}^+ + \text{monopole (+ pions)}.$$

Then protons can decay with dramatic rates at low energies.

For a review and more examples of circumstances in which monopoles may appear see e.g. [5].

5.2 Yang-Mills-Higgs Model and the Bogomolny Equations

Let us consider the Minkowski space $\mathbb{R}^{3,1}$ with the metric diag$(1,1,1,-1)$, a Lie-algebra valued gauge potential $A = A_\mu \mathrm{d}x^\mu$ and the Yang-Mills field $F = \mathrm{d}A + A \wedge A$ with field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ and a real scalar field Φ belonging to the adjoint representation of the gauge group. Furthermore, we will consider static solutions for A and Φ, i.e. independent on x^4 and $A_4 = 0$. In the following we will specify to the gauge groups $U(1)$ and $SU(2)$. We have the Yang-Mills-Higgs Lagrangian density [51]

$$\mathcal{L} = -\frac{1}{4}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu} + 2D^\mu\Phi D_\mu\Phi) - \frac{\gamma}{4}(|\Phi|^2 - 1)^2, \tag{5.2}$$

where $D_\mu = \partial_\mu + [A_\mu, \cdot]$, $|\Phi|^2 = -\mathrm{tr}(\Phi^2)$ and $\gamma \geq 0$ is a real constant.

In the Bogomolny-Prasad-Sommerfield limit [7, 40], where the self-interaction part of the Higgs field vanishes ($\gamma = 0$), the equations of motion reduce to the first order Bogomolny equations ($a, b, c, \ldots = 1, 2, 3$)

$$F_{ab} = \epsilon_{abc} D_c \Phi = \epsilon_{abc}(\partial_c \Phi + [A_c, \Phi]), \tag{5.3}$$

which minimize the energy functional

$$\begin{aligned} E &= -\frac{1}{4}\int \mathrm{d}^3x \ \mathrm{tr}(F_{ab}F_{ab} + 2D_a\Phi D_a\Phi) \\ &= -\frac{1}{4}\int \mathrm{d}^3x \ \mathrm{tr}(F_{ab} \mp \epsilon_{abc}D_c\Phi)(F_{ab} \mp \epsilon_{abd}D_d\Phi) \mp \int \mathrm{d}^3x \ \mathrm{tr}(\frac{1}{2}\epsilon_{abc}F_{bc}D_c\Phi) \\ &= -\frac{1}{4}\int \mathrm{d}^3x \ \mathrm{tr}(F_{ab} \mp \epsilon_{abc}D_c\Phi)(F_{ab} \mp \epsilon_{abd}D_d\Phi) \pm 4\pi Q \geq 4\pi|Q|, \end{aligned} \tag{5.4}$$

where $Q \in \mathbb{Z}$ is the magnetic charge. Solutions to (5.3) with $Q = n$ are interpreted as n monopoles.

The Bogomolny equations (5.3) can be be obtained as the compatibility condition of two linear differential equations from a parametric Riemann-Hilbert problem as we will see in section 5.3. Let us introduce the complex coordinates

$$z^1 = x^1 + \mathrm{i}x^2 \qquad \text{and} \qquad z^2 = x^3 - \mathrm{i}x^4 \tag{5.5}$$

as well as

$$A_{z^1} = \frac{1}{2}(A_1 - \mathrm{i}A_2),\ A_{\bar{z}^1} = \frac{1}{2}(A_1 + \mathrm{i}A_2),\ A_{z^2} = \frac{1}{2}(A_3 + \mathrm{i}\Phi) \text{ and } A_{\bar{z}^2} = \frac{1}{2}(A_3 - \mathrm{i}\Phi). \tag{5.6}$$

We define $D'_a = \partial_a + A_a$ for $a = 1, 2, 3$ and $D'_4 = \partial_4 + \Phi$ and suppose that (A_a, Φ) are independent of x^4. Then the Bogomolny equations (5.3) read

$$[D'_{z^1}, D'_{z^2}] = 0, \qquad [D'_{\bar{z}^1}, D'_{\bar{z}^2}] = 0, \qquad [D'_{z^1}, D'_{\bar{z}^1}] + [D'_{z^2}, D'_{\bar{z}^2}] = 0. \tag{5.7}$$

Note that the Bogomolny equations (5.3) can be obtained by dimensional reduction from the self-dual Yang-Mills equations (3.11) in the Euclidean space $\mathbb{R}^4$.

5.3 Riemann-Hilbert Problem and Reality Condition

The extended complex plane $\mathbb{C}P^1$ can be covered by two coordinate patches U_+ and U_- with

$$\mathbb{C}P^1 = U_+ \cup U_-, \qquad U_+ = \mathbb{C}P^1 \setminus \{\infty\}, \qquad U_- = \mathbb{C}P^1 \setminus \{0\}, \tag{5.8}$$

and coordinates λ and $\tilde{\lambda}$ on U_+ and U_-, respectively, which are related by $\lambda = \tilde{\lambda}^{-1}$ on the intersection $U_+ \cap U_-$. We also introduce the complex coordinates

$$\eta = z^1 - 2\lambda x^3 - \lambda^2 \bar{z}^1 \quad \text{and} \quad \tilde{\eta} = -\tilde{\lambda}^2 z^1 + 2\tilde{\lambda} x^3 + \bar{z}^1 \tag{5.9}$$

on $V_+ = \mathbb{R}^2 \times U_+$ and $V_- = \mathbb{R}^2 \times U_-$, respectively, which are two patches covering the so-called mini-twistor space $\mathcal{T} \cong \mathcal{O}(2)$ (see e.g. [30] and references therein).

Let us consider a transition function $f_{+-}(x^1, x^2, x^3, \lambda) = f_{+-}(\eta, \lambda)|_{S^1_x}$ of a holomorphic vector bundle over the mini-twistor space $\mathcal{T}$ restricted to a circle $S^1_x \subset U_+ \cap U_- \subset \mathbb{C}P^1 \hookrightarrow \mathcal{T}$. Since f_{+-} depends holomorphically on η and λ, we find

$$(\partial_{\bar{z}^1} - \lambda \partial_{z^2})\ f_{+-}(\eta, \lambda) = (\partial_{\bar{z}^2} + \lambda \partial_{z^1})\ f_{+-}(\eta, \lambda) = 0. \tag{5.10}$$

One may try to split $f_{+-}(x^a, \lambda)$ as

$$f_{+-}(x^a, \lambda) = \psi_+^{-1}(x^a, \lambda)\ \psi_-(x^a, \lambda), \tag{5.11}$$

where $\psi_+(x^a, \lambda)$ and $\psi_-(x^a, \lambda)$ for $a = 1, 2, 3$ are boundary values of holomorphic functions in λ on U_+ and U_-, respectively. That means that $\psi_+(x^a, \lambda)$ is holomorphic in λ on a subset of U_+ around zero and $\psi_-(x^a, \lambda)$ is holomorphic in [2] $\tilde{\lambda} = \lambda^{-1}$ on a subset of U_- around infinity, where the intersection of this two subsets is non-empty and their union is equal to $\mathbb{C}P^1$. Inserting (5.11) into (5.10), we obtain

$$\begin{aligned}
\psi_+(x^a, \lambda)\ (\partial_{\bar{z}^1} - \lambda \partial_{z^2})\ \psi_+^{-1}(x^a, \lambda) &= \psi_-(x^a, \lambda)\ (\partial_{\bar{z}^1} - \lambda \partial_{z^2})\ \psi_-^{-1}(x^a, \lambda), \\
\psi_+(x^a, \lambda)\ (\partial_{\bar{z}^2} + \lambda \partial_{z^1})\ \psi_+^{-1}(x^a, \lambda) &= \psi_-(x^a, \lambda)\ (\partial_{\bar{z}^2} + \lambda \partial_{z^1})\ \psi_-^{-1}(x^a, \lambda).
\end{aligned} \tag{5.12}$$

Due to the fact that a function, which is holomorphic on the entire $\mathbb{C}P^1$, is constant (Liouville's theorem) we conclude that (5.12) is linear in λ. Hence, we can define

$$\begin{aligned}
A_{\bar{z}^1} - \lambda A_{z^2} &= \psi_\pm(x^a, \lambda)\ (\partial_{\bar{z}^1} - \lambda \partial_{z^2})\ \psi_\pm^{-1}(x^a, \lambda) \quad \text{and} \\
A_{\bar{z}^2} + \lambda A_{z^1} &= \psi_\pm(x^a, \lambda)\ (\partial_{\bar{z}^2} + \lambda \partial_{z^1})\ \psi_\pm^{-1}(x^a, \lambda).
\end{aligned} \tag{5.13}$$

This can be written as the linear system

$$(D'_{\bar{z}^1} - \lambda D'_{z^2})\ \psi_\pm(x^a, \lambda) = 0 \quad \text{and} \quad (D'_{\bar{z}^2} + \lambda D'_{z^1})\ \psi_\pm(x^a, \lambda) = 0, \tag{5.14}$$

which has (5.7) as the compatibility conditions. Thus, A_a and Φ, obtained from

$$\frac{1}{2}(A_1 + \mathrm{i}A_2) = \psi_+(x^a, \lambda)\ \partial_{\bar{z}^1}\ \psi_+^{-1}(x^a, \lambda)|_{\lambda=0},\ \frac{1}{2}(A_3 - \mathrm{i}\Phi) = \psi_+(x^a, \lambda)\ \partial_{\bar{z}^2}\ \psi_+^{-1}(x^a, \lambda)|_{\lambda=0} \tag{5.15}$$

[2] Note that this notation is a bit sloppy, but we hope that it is clear what we mean.

and

$$\frac{1}{2}(A_1 - \mathrm{i}A_2) = \psi_-(x^a,\lambda)\ \partial_{z^1}\ \psi_-^{-1}(x^a,\lambda)|_{\lambda=\infty},\ \frac{1}{2}(A_3 + \mathrm{i}\Phi) = \psi_-(x^a,\lambda)\ \partial_{z^2}\ \psi_-^{-1}(x^a,\lambda)|_{\lambda=\infty}, \tag{5.16}$$

satisfy the Bogomolny equation (5.3).

The gauge fields A_a for $a \in \{1,2,3\}$ and the Higgs field Φ obtained from (5.15) and (5.16) are anti-hermitian, if

$$\psi_+^\dagger(x^a, -\bar{\lambda}^{-1}) = \psi_-^{-1}(x^a,\lambda). \tag{5.17}$$

In order to conform to (5.17) we demand that $f_{+-}(x^1,x^2,x^3,\lambda)$ has positive determinant and fulfils the following reality condition:

$$f_{+-}^\dagger(x^1,x^2,x^3,-\bar{\lambda}^{-1}) = f_{+-}(x^1,x^2,x^3,\lambda). \tag{5.18}$$

Inserting (5.11) into (5.18), we find

$$\psi_-(x^a,\lambda)\ \psi_+^\dagger(x^a,-\bar{\lambda}^{-1}) = \psi_+(x^a,\lambda)\ \psi_-^\dagger(x^a,-\bar{\lambda}^{-1}), \tag{5.19}$$

where the right hand side is holomorphic on U_+ and the left hand side holomorphic on U_- and therefore on the entire $\mathbb{C}P^1$. We conclude that the matrix in (5.19) is independent of λ. Furthermore, the matrix in (5.19) is hermitian

$$(\psi_-(x^a,\lambda)\ \psi_+^\dagger(x^a,-\bar{\lambda}^{-1}))^\dagger|_{\lambda=\infty} = \psi_+(x^a,\lambda)\ \psi_-^\dagger(x^a,-\bar{\lambda}^{-1})|_{\lambda=0}. \tag{5.20}$$

We have

$$\psi_-(x^a,\lambda)\ \psi_+^\dagger(x^a,-\bar{\lambda}^{-1}) = g(x^a)\ g^\dagger(x^a), \tag{5.21}$$

if and only if the matrix in (5.19) is positive definite, i.e. has only positive eigenvalues. If $g\ g^\dagger \neq \mathbf{1}$ then we can perform a non-unitary gauge transformation

$$\psi_\pm(x^a,\lambda) \mapsto \psi_\pm^g(x^a,\lambda) = g^{-1}(x^a)\ \psi_\pm(x^a,\lambda) \tag{5.22}$$

inducing

$$A_a \mapsto A_a^g = g^{-1}A_a g + g^{-1}\partial_a g \qquad \text{and} \qquad \Phi \mapsto \Phi^g = g^{-1}\Phi g \tag{5.23}$$

and find that $\psi_\pm^g(x^a,\lambda)$ satisfies (5.17). Obviously $f_{+-}(\eta,\lambda)$ remains unchanged under the gauge transformation (5.22).

Note that positivity of the determinant of $f_{+-}(\eta,\lambda)$ is necessary for (5.17) as well as for (5.21) (and thus also for (5.22)) but not sufficient. Suppose that $\det(f_{+-}(\eta,\lambda)) = -1$. Then from (5.11) it follows that

$$\det(\psi_+(x^a,\lambda)) = -\det(\psi_-(x^a,\lambda)). \tag{5.24}$$

Thus, $\det(\psi_\pm)$ is independent of λ. Substituting (5.24) into (5.21) and performing the determinant, we end up with

$$|\det(g)|^2 = -|\det(\psi_\pm)|^2, \tag{5.25}$$

which is obviously a contradiction since $\psi_\pm$ must have non-zero determinant.

5.4 Dirac Monopoles

Now we want to present the famous Dirac monopole solution [15] obtained via the splitting approach (see section 5.3). Then we will consider the case of two Dirac monopoles which can be easily generalized to a Dirac multi-monopole solution. In our discussion we follow [30].

One Dirac monopole. The gauge potential of the Dirac monopole is (globally) defined as two local potentials on two patches covering the base space of the $U(1)$ bundle over $\mathbb{R}^3 \setminus \{0\}$, which are related by some transition function on the intersection of these patches. In the twistor approach, we will consider the holomorphic transition function (see section 5.3) $f^D_{-+} = (f^D_{+-})^{-1} = \frac{\eta}{\lambda}$ of the line bundle over $\mathcal{T}$, which is well defined in $\lambda \in U_+ \cap U_-$ apart from $\eta \neq 0$. Let us redenote z^1 by y. We have

$$\eta = y - 2\lambda x^3 - \lambda^2 \bar{y} = -(\lambda - \lambda_0)\,(\lambda + \bar{\lambda}_0^{-1})\,\bar{y}, \tag{5.26}$$

where $\lambda_0 = -\frac{y}{\xi_S} = -\frac{\xi_N}{\bar{y}}$ with $\xi_S = r - x^3$, $\xi_N = r + x^3$ and $r^2 = y\bar{y} + x^3x^3$. Assume $|\lambda_0| \leq 1$ or replace λ_0 by $-\bar{\lambda}_0^{-1}$ otherwise. We consider the circle S^1 in $\mathbb{C}P^1$ defined as

$$S^1 \;=\; \begin{cases} \{\lambda \in \mathbb{C}P^1 \mid \quad |\lambda| = 1 \quad \text{if} \quad |\lambda_0| < 1 < |\lambda_0|^{-1}\} \\ \{\lambda \in \mathbb{C}P^1 \mid \quad |\lambda - \epsilon\lambda_0| = 1 \text{ with } 0 < \epsilon < 1 \quad \text{if} \quad |\lambda_0| = 1\} \end{cases} \tag{5.27}$$

on which η is invertible. Note that in the second case of (5.27) we have $|\lambda_0 - \epsilon\lambda_0| = 1 - \epsilon < 1 < 1 + \epsilon = |-\bar{\lambda}_0^{-1} - \epsilon\lambda_0|$. Thus, the circle S^1 separates the points $\lambda = 0$ and $\lambda = \lambda_0$ from the points $\lambda = \infty$ and $\lambda = -\bar{\lambda}_0^{-1}$. We will restrict f^D_{-+} to S^1.

One can find a splitting according to (5.11)

$$f^D_{-+} = \underbrace{\left(\xi_S^{\frac{1}{2}} + \lambda^{-1} y \xi_S^{-\frac{1}{2}}\right)}_{(\psi^S_-)^{-1}} \underbrace{\left(\xi_S^{\frac{1}{2}} - \lambda \xi_S^{-\frac{1}{2}} \bar{y}\right)}_{\psi^S_+} = \underbrace{\xi_S^{\frac{1}{2}}\left(1 - \lambda^{-1}\lambda_0\right)}_{(\psi^S_-)^{-1}} \underbrace{\xi_S^{\frac{1}{2}}\left(1 + \lambda\bar{\lambda}_0\right)}_{\psi^S_+}. \tag{5.28}$$

We have that $(\psi^S_+)^{-1} = \xi_S^{-\frac{1}{2}} \sum_{k=0}^{\infty} (-\bar{\lambda}_0)^k \lambda^k$ is holomorphic in λ on $\{\lambda \in \mathbb{C}P^1 \mid |\lambda| < |\lambda_0|^{-1}\} \subset U_+$ and $\psi^S_- = \xi_S^{-\frac{1}{2}} \sum_{k=0}^{\infty} \lambda_0^k \lambda^{-k}$ in λ^{-1} on $\{\lambda \in \mathbb{C}P^1 \mid |\lambda| > |\lambda_0|\} \subset U_-$. Another splitting (again according to (5.11)) would be

$$f^D_{-+} = \underbrace{\left(\bar{y}\xi_N^{-\frac{1}{2}} + \lambda^{-1} \xi_N^{\frac{1}{2}}\right)}_{(\psi^N_-)^{-1}} \underbrace{\left(\xi_N^{-\frac{1}{2}} y - \lambda \xi_N^{\frac{1}{2}}\right)}_{\psi^N_+} = \underbrace{\bar{y}\xi_N^{-\frac{1}{2}}\left(1 - \lambda^{-1}\lambda_0\right)}_{(\psi^N_-)^{-1}} \underbrace{y\xi_N^{-\frac{1}{2}}\left(1 + \lambda\bar{\lambda}_0\right)}_{\psi^N_+}. \tag{5.29}$$

Here we have again supposed that $|\lambda_0| \leq 1$. If this is not true, we have to exchange $\xi_S \leftrightarrow \xi_N$ in (5.28) and (5.29). Using (5.15) and (5.16), we obtain the following gauge potentials and scalar field from (5.28):

$$A^S_1 = -\frac{\mathrm{i}x^2}{2r(r - x^3)}, \quad A^S_2 = \frac{\mathrm{i}x^1}{2r(r - x^3)}, \quad A^S_3 = 0 \quad \text{and} \quad \Phi^S = \frac{\mathrm{i}}{2r}, \tag{5.30}$$

which are regular on $\mathbb{R}^3_S = \mathbb{R}^3 \setminus \{x^1 = x^2 = 0, x^3 \geq 0\}$. From (5.29) we obtain

$$A^N_1 = \frac{\mathrm{i}x^2}{2r(r + x^3)}, \quad A^N_2 = -\frac{\mathrm{i}x^1}{2r(r + x^3)}, \quad A^N_3 = 0 \quad \text{and} \quad \Phi^S = \frac{\mathrm{i}}{2r} \tag{5.31}$$

on $\mathbb{R}^3_N = \mathbb{R}^3 \setminus \{x^1 = x^2 = 0, x^3 \le 0\}$. On the intersection $\mathbb{R}^3_S \cap \mathbb{R}^3_N$, the gauge potentials and the scalar fields are related by

$$A^N = f^D_{NS}\ A^S\ (f^D_{NS})^{-1} + f^D_{NS}\ \mathrm{d}(f^D_{NS})^{-1} \quad \text{and} \quad \Phi^N = f^D_{NS}\ \Phi^S\ (f^D_{NS})^{-1} \tag{5.32}$$

with the transition function $f^D_{NS} = (\frac{y}{\bar{y}})^{\frac{1}{2}}$.

Two Dirac monopoles. We will use the notation $\eta_i = y_i - 2\lambda x^3_i - \lambda^2 \bar{y}_i = -(\lambda - \lambda_i)\ (\lambda + \bar{\lambda}_i^{-1})\ \bar{y}_i$, where $\vec{x}_i = (x^1 - b^1_i, x^2 - b^2_i, x^3 - b^3_i)$ and $\vec{b}_i = (b^1_i, b^2_i, b^3_i)$ are some constant vectors, referring to the position of the monopoles in $\mathbb{R}^3$. Furthermore, we have $\lambda_i = -\frac{y_i}{\xi_{Si}} = -\frac{\xi_{Ni}}{\bar{y}_i}$ with $\xi_{Si} = r_i - x^3_i$, $\xi_{Ni} = r_i + x^3_i$ and $r_i^2 = y_i \bar{y}_i + x^3_i x^3_i$. Let us consider the holomorphic function $f^{(2)D}_{-+} = \frac{\eta_1 \eta_2}{\lambda^2}$ and restrict it to the contour which separates the points $\lambda = 0, \lambda_1, \lambda_2$ from the points $\lambda = -\bar{\lambda}_1^{-1}, -\bar{\lambda}_2^{-1}, \infty$. Then with (5.28) and (5.29) we have

$$f^{(2)D}_{-+} = \big(\psi^S_-(\vec{x}_1)\ \psi^S_-(\vec{x}_2)\big)^{-1}\ \ \psi^S_+(\vec{x}_1)\ \psi^S_+(\vec{x}_2) = \big(\psi^N_-(\vec{x}_1)\ \psi^N_-(\vec{x}_2)\big)^{-1}\ \ \psi^N_+(\vec{x}_1)\ \psi^N_+(\vec{x}_2). \tag{5.33}$$

Then the gauge potential and the scalar field in the two-monopole case are given in terms of (5.30) and (5.31) by

$$A^{(2)S} = A^S(\vec{x}_1) + A^S(\vec{x}_2) \quad \text{and} \quad \Phi^{(2)S} = \Phi^S(\vec{x}_1) + \Phi^S(\vec{x}_2) \tag{5.34}$$

on $\mathbb{R}^3_S = \mathbb{R}^3 \setminus (\{x^1_1 = x^2_1 = 0, x^3_1 \ge 0\} \cup \{x^1_2 = x^2_2 = 0, x^3_2 \ge 0\})$ as well as

$$A^{(2)N} = A^N(\vec{x}_1) + A^N(\vec{x}_2) \quad \text{and} \quad \Phi^{(2)N} = \Phi^N(\vec{x}_1) + \Phi^N(\vec{x}_2) \tag{5.35}$$

on $\mathbb{R}^3_N = \mathbb{R}^3 \setminus (\{x^1_1 = x^2_1 = 0, x^3_1 \le 0\} \cup \{x^1_2 = x^2_2 = 0, x^3_2 \le 0\})$ related on the intersection $\mathbb{R}^3_S \cap \mathbb{R}^3_N$ by (5.32) with the transition function $f^{(2)D}_{NS} = \left(\frac{y_1 y_2}{\bar{y}_1 \bar{y}_2}\right)^{\frac{1}{2}}$. This gives a global description of the gauge potential as long as the two monopoles do not sit on a line parallel to the x^3-axis, i.e. $b^1_1 \ne b^1_2$ and $b^2_1 \ne b^2_2$. In the case of monopoles sitting on such a line, the gauge potential of the Dirac two-monopole solution and the scalar field may be defined on three patches covering $\mathbb{R}^3 \setminus \{\vec{b}_1, \vec{b}_2\}$. If we choose $b_1 > b_2$, we additionally have

$$A^{(2)M} = A^S(\vec{x}_1) + A^N(\vec{x}_2) \quad \text{and} \quad \Phi^{(2)M} = \Phi^S(\vec{x}_1) + \Phi^N(\vec{x}_2) \tag{5.36}$$

on $\mathbb{R}^3_M = \mathbb{R}^3 \setminus \{x^1_1 = x^2_1 = 0, x^3 \le b^3_2, x^3 \ge b^3_1\}$.

Dirac multi-monopoles. We can obtain Dirac n-monopole solutions via splitting of the holomorphic function $f^{(n)D}_{-+} = \prod_{i=1}^n \frac{\eta_i}{\lambda}$. Then we have $A^{(n)S} = \sum_{i=1}^n A^S(\vec{x}_i)$ and $\Phi^{(n)S} = \sum_{i=1}^n \Phi^S(\vec{x}_i)$ on $\mathbb{R}^3_S = \mathbb{R}^3 \setminus (\bigcup_{i=1}^n \{x^1_i = x^2_i = 0, x^3_i \ge 0\})$ as well as $A^{(n)N} = \sum_{i=1}^n A^N(\vec{x}_i)$ and $\Phi^{(n)N} = \sum_{i=1}^n \Phi^N(\vec{x}_i)$ on $\mathbb{R}^3_N = \mathbb{R}^3 \setminus (\bigcup_{i=1}^n \{x^1_i = x^2_i = 0, x^3_i \le 0\})$. But we have to be careful with solutions where some monopoles are aligned on the same line parallel to the x^3-axis. Then we may need more coverings of $\mathbb{R}^3 \setminus \{\vec{b}_1, \ldots, \vec{b}_n\}$ for a global description of the multi-monopole solution.

5.5 Wu-Yang $SU(2)$ Monopoles

The spherically-symmetric Wu-Yang $SU(2)$ monopole [57] was first thought to be non-abelian, but it turned out (see [51, 58]) that it is just the abelian $U(1)$ Dirac monopole embedded in $SU(2)$. It is a static solution of the equation of motion in Yang-Mills-Higgs theory [1]. Furthermore, it satisfies the Bogomolny equation (5.3), but has infinite energy. In the following we will recall the results for the Wu-Yang monopole solution first.

One Wu-Yang monopole. We choose the Atiyah-Ward $\mathcal{A}_1$ ansatz [4] for the transition function (see section 5.3) of a rank two holomorphic vector bundle over $\mathcal{T}$ [30]

$$f_{+-}^{(1)} = \begin{pmatrix} \rho^{(1)} & \lambda^{-1} \\ -\lambda & 0 \end{pmatrix}, \tag{5.37}$$

where $\rho^{(1)} = \frac{\lambda}{\eta}$. We restrict $f_{+-}^{(1)}$ to the circle S^1 given in (5.27). This matrix-valued function $f_{+-}^{(1)}$ is easy to factorize according to (5.11), where the obtained ψ_+ and ψ_- are holomorphic on the entire U_+ and U_-, respectively. Inserting ψ_+ in (5.15) and ψ_- in (5.16), we find

$$A_a = -\epsilon_{abc}\,\frac{\sigma_c}{2\mathrm{i}}\;\rho_0^{-1}\;\partial_b\rho_0 \qquad \text{and} \qquad \Phi = -\frac{\sigma_a}{2\mathrm{i}}\;\rho_0^{-1}\;\partial_a\rho_0, \tag{5.38}$$

where $\rho_0 = \frac{1}{2r}$. For a detailed calculation see [30].

Two Wu-Yang monopoles. We want to consider a matrix Riemann-Hilbert problem for two Wu-Yang monopoles. Therefore, we need to find some transition function $f_{+-}^{(2)}$ of a holomorphic rank two vector bundle over the minitwistor space $\mathcal{T}$. We have a look on the Atiyah-Ward $\mathcal{A}_n$ ansatz [4] for $n = 2$,

$$f_{+-}^{(n)} = \begin{pmatrix} \rho^{(n)} & \lambda^{-n} \\ -\lambda^n & 0 \end{pmatrix}, \tag{5.39}$$

where $\rho^{(n)}$, as a function depending on η and λ only, is holomorphic in its variables and satisfies the condition

$$\bar{\rho}^{(n)}(x^1, x^2, x^3, -\bar{\lambda}^{-1}) = \rho^{(n)}(x^1, x^2, x^3, \lambda). \tag{5.40}$$

One can easily see that the Atiyah-Ward ansatz (5.39) does not satisfy (5.18) for even n. We need to find a modification of (5.39) for $n = 2$, which satisfies (5.18) and has positive determinant, such as

$$f_{+-}^{(2)} = k_+\,\begin{pmatrix} \rho^{(2)} & \lambda^{-2} \\ -\lambda^2 & 0 \end{pmatrix}\,k_-, \tag{5.41}$$

where k_+ and k_- are two invertible matrix-valued functions holomorphic in (η, λ) and $(\tilde{\eta}, \tilde{\lambda})$, respectively. Furthermore, (5.19) has to be positive definite. From the twistor argument (see e.g. [30] and references therein), we know that we will not change the configuration in (5.41) by multiplying with $k_\pm$. Moreover, we will obtain the same gauge fields as if we consider (5.39) as the holomorphic transition function. We will see that in the following.

The task of splitting $f_{+-}^{(2)}$ given by (5.41) according to (5.11) reduces to the factorization problem

$$\psi'_+(x^1, x^2, x^3, \lambda)\,\begin{pmatrix} \rho^{(2)} & \lambda^{-2} \\ -\lambda^2 & 0 \end{pmatrix} = \psi'_-(x^1, x^2, x^3, \lambda), \tag{5.42}$$

where ψ'_+ and ψ'_- are boundary values of holomorphic functions in λ on U_+ and U_-, respectively.

We can expand $\rho^{(2)}$ in a Laurent series in λ

$$\rho^{(2)} = \underbrace{\sum_{k=-1}^{-\infty} \rho_k \lambda^k}_{\rho_-} + \rho_0 + \underbrace{\sum_{k=1}^{\infty} \rho_k \lambda^k}_{\rho_+} \tag{5.43}$$

since $\rho^{(2)}$ should be analytic on S^1. For the sake of clarity we omitted the superscript $^{(2)}$ in the coefficients of $\rho^{(2)}$. From the holomorphicity of $\rho^{(2)}$ in η_i, we get

$$(2\partial_{\bar{y}} - \lambda\partial_3)\rho^{(2)} = (\partial_3 + 2\lambda\partial_y)\rho^{(2)} = 0 \tag{5.44}$$

which implies the following recursion relations for the coefficients in the Laurent series of $\rho^{(2)}$:

$$2\partial_{\bar{y}}\rho_{k+1} = \partial_3\rho_k = -2\partial_y\rho_{k-1}. \tag{5.45}$$

Supposing that the entries in ψ'_+ (ψ'_-) are functions analytic in λ ($\tilde{\lambda}$) on the entire U_+ (U_-), we can expand them in a Taylor series in λ (λ^{-1}). From (5.42) we obtain conditions on the coefficients in the Taylor series of the entry functions of ψ'_+ [11]. These conditions still leave four of the six coefficients arbitrary. In addition we need to demand $\det(\psi'_+) \neq 0$ for invertibility reasons. The remaining freedom is due to some gauge invariance. We find

$$\psi'_+(x,\lambda) = \begin{pmatrix} a_0 + a_1\lambda + a_2\lambda^2 & a_0(\rho_+ - \rho_1\lambda)\lambda^{-2} + a_1\rho_+\lambda^{-1} + a_2(\rho_0 + \rho_+) \\ c_0 + c_1\lambda + c_2\lambda^2 & c_0(\rho_+ - \rho_1\lambda)\lambda^{-2} + c_1\rho_+\lambda^{-1} + c_2(\rho_0 + \rho_+) \end{pmatrix}, \tag{5.46}$$

where $a_1\rho_0 = -(a_0\rho_1 + a_2\rho_{-1})$, $c_1\rho_0 = -(c_0\rho_1 + c_2\rho_{-1})$. The coefficients a_0, a_2, c_0 and c_2 remain arbitrary but need to satisfy $a_0c_2 - a_2c_0 \neq 0$. Using (5.42) we obtain

$$\psi'_-(x,\lambda) = \begin{pmatrix} a_0(\rho_- + \rho_0) + a_1\rho_-\lambda + a_2(\rho_- - \rho_{-1}\lambda^{-1})\lambda^2 & a_2 + a_1\lambda^{-1} + a_0\lambda^{-2} \\ c_0(\rho_- + \rho_0) + c_1\rho_-\lambda + c_2(\rho_- - \rho_{-1}\lambda^{-1})\lambda^2 & c_2 + c_1\lambda^{-1} + c_0\lambda^{-2} \end{pmatrix}. \tag{5.47}$$

Thus, we have found a splitting $f^{(2)}_{+-} = \psi_+^{-1}\psi_-$ with

$$\psi_+ = \psi'_+\, k_+^{-1} \qquad \text{and} \qquad \psi_- = \psi'_-\, k_-. \tag{5.48}$$

Then we know from section 5.3 that A_a for $a \in \{1,2,3\}$ and Φ, obtained by using the formulas

$$\begin{aligned} A_1 + \mathrm{i}A_2 &= 2\left(\psi'_+\ \partial_{\bar{y}}\ (\psi'_+)^{-1}\right)|_{\lambda=0}, & A_3 - \mathrm{i}\Phi &= \psi'_+(\lambda)\ \partial_3\ (\psi'_+)^{-1}(\lambda)|_{\lambda=0}, \\ A_1 - \mathrm{i}A_2 &= 2\left(\psi'_-\ \partial_y\ (\psi'_-)^{-1}\right)|_{\lambda=\infty} \quad \text{and} & A_3 + \mathrm{i}\Phi &= \psi'_-\ \partial_3\ (\psi'_-)^{-1}|_{\lambda=\infty}, \end{aligned} \tag{5.49}$$

satisfy the Bogomolny equation (5.3).

To become more concrete we choose a frame where the two monopoles sit on the x^3-axis and consider the function

$$\rho^{(2)} = \frac{2\ (b_1 - b_2)\ \lambda^2}{\eta_1\ \eta_2} = \frac{\lambda}{\eta_1} - \frac{\lambda}{\eta_2}. \tag{5.50}$$

We calculate

$$\begin{aligned}\rho_0 &= \oint_{|\lambda|=1} \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}\ \lambda}\ \rho^{(2)} = \frac{1}{2r_1} - \frac{1}{2r_2} \text{ and} \\ \rho_1 &= \oint_{|\lambda|=1} \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}\ \lambda^2}\ \rho^{(2)} = \frac{x^3-b_1}{2\ r_1\ y} - \frac{x^3-b_2}{2\ r_2\ y} \\ &= -\bar{\rho}_{-1}. \end{aligned} \tag{5.51}$$

The last line in (5.51) is true due to condition (5.40). We restrict $f_{+-}^{(2)}$ to a circle on which η_i for $i=1,2$ is invertible. Let us consider $k_- = \mathbf{1}_2$ and k_+ such that

$$f_{+-}^{(2)} = \begin{pmatrix} \rho^{(2)} & \lambda^{-2} \\ \lambda^2 & 2(\rho^{(2)})^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2\lambda^2(\rho^{(2)})^{-1} & 1 \end{pmatrix} \begin{pmatrix} \rho^{(2)} & \lambda^{-2} \\ -\lambda^2 & 0 \end{pmatrix}, \tag{5.52}$$

where $\frac{2\lambda^2}{\rho^{(2)}} = \frac{\eta_1\ \eta_2}{b_1-b_2}$ is a holomorphic function on V_+. This choice (5.52) satisfies (5.18) and has positive determinant. Furthermore, we choose $a_0 = \sqrt{\frac{\rho_0}{\rho_d}}$, $a_1 = -\frac{\rho_1}{\sqrt{\rho_0\rho_d}}$, $a_2 = 0$, $c_0 = 0$, $c_1 = -\frac{\rho_{-1}}{\sqrt{\rho_0\rho_d}}$ and $c_2 = \sqrt{\frac{\rho_0}{\rho_d}}$, where $\rho_d = \rho_0^2 - \rho_1\ \rho_{-1}$. Then we insert ψ_+ and ψ_- into (5.49) and obtain, using (5.45),

$$\begin{aligned}
A_1 &= \begin{pmatrix} -\frac{\mathrm{i}}{2}\left(\frac{\partial_2\ \rho_0}{\rho_0} - \frac{\partial_2\ \rho_d}{\rho_d}\right) & -\frac{\rho_1}{2\rho_0}\left(\frac{\partial_3\ \rho_1}{\rho_1} - \frac{\partial_3\ \rho_d}{\rho_d}\right) \\ \frac{\rho_{-1}}{2\rho_0}\left(\frac{\partial_3\ \rho_{-1}}{\rho_{-1}} - \frac{\partial_3\ \rho_d}{\rho_d}\right) & \frac{\mathrm{i}}{2}\left(\frac{\partial_2\ \rho_0}{\rho_0} - \frac{\partial_2\ \rho_d}{\rho_d}\right) \end{pmatrix}, \\
A_2 &= \begin{pmatrix} \frac{\mathrm{i}}{2}\left(\frac{\partial_2\ \rho_0}{\rho_0} - \frac{\partial_2\ \rho_d}{\rho_d}\right) & \frac{\mathrm{i}\rho_1}{2\rho_0}\left(\frac{\partial_3\ \rho_1}{\rho_1} - \frac{\partial_3\ \rho_d}{\rho_d}\right) \\ \frac{\mathrm{i}\rho_{-1}}{2\rho_0}\left(\frac{\partial_3\ \rho_{-1}}{\rho_{-1}} - \frac{\partial_3\ \rho_d}{\rho_d}\right) & -\frac{\mathrm{i}}{2}\left(\frac{\partial_2\ \rho_0}{\rho_0} - \frac{\partial_2\ \rho_d}{\rho_d}\right) \end{pmatrix}, \\
A_3 &= \begin{pmatrix} 0 & \frac{\rho_1}{2\rho_0}\left(\frac{(\partial_1-\mathrm{i}\partial_2)\ \rho_1}{\rho_1} - \frac{(\partial_1-\mathrm{i}\partial_2)\ \rho_d}{\rho_d}\right) \\ -\frac{\rho_{-1}}{2\rho_0}\left(\frac{(\partial_1+\mathrm{i}\partial_2)\ \rho_{-1}}{\rho_{-1}} - \frac{(\partial_1+\mathrm{i}\partial_2)\ \rho_d}{\rho_d}\right) & 0 \end{pmatrix} \quad \text{and} \\
\Phi &= \begin{pmatrix} -\frac{\mathrm{i}}{2}\left(\frac{\partial_3\ \rho_0}{\rho_0} - \frac{\partial_3\ \rho_d}{\rho_d}\right) & -\frac{\mathrm{i}\rho_1}{2\rho_0}\left(\frac{(\partial_1-\mathrm{i}\partial_2)\ \rho_1}{\rho_1} - \frac{(\partial_1-\mathrm{i}\partial_2)\ \rho_d}{\rho_d}\right) \\ -\frac{\mathrm{i}\rho_{-1}}{2\rho_0}\left(\frac{(\partial_1+\mathrm{i}\partial_2)\ \rho_{-1}}{\rho_{-1}} - \frac{(\partial_1+\mathrm{i}\partial_2)\ \rho_d}{\rho_d}\right) & \frac{\mathrm{i}}{2}\left(\frac{\partial_3\ \rho_0}{\rho_0} - \frac{\partial_3\ \rho_d}{\rho_d}\right) \end{pmatrix}.
\end{aligned} \tag{5.53}$$

This coincides with the formula given in [13] and can be written in compact form

$$\begin{aligned}
A_a &= \frac{-\mathrm{i}\rho_d}{2\rho_0} \begin{pmatrix} \eta^3_{ab}\ \partial_b\left(\frac{\rho_0}{\rho_d}\right) & -(\eta^1_{ab} - \mathrm{i}\eta^2_{ab})\ \partial_b\left(\frac{\rho_1}{\rho_d}\right) \\ -(\eta^1_{ab} + \mathrm{i}\eta^2_{ab})\ \partial_b\left(\frac{\rho_{-1}}{\rho_d}\right) & -\eta^3_{ab}\ \partial_b\left(\frac{\rho_0}{\rho_d}\right) \end{pmatrix} \quad \text{and} \\
\Phi &= \frac{-\mathrm{i}\rho_d}{2\rho_0} \begin{pmatrix} \eta^3_{4b}\ \partial_b\left(\frac{\rho_0}{\rho_d}\right) & -(\eta^1_{4b} - \mathrm{i}\eta^2_{4b})\ \partial_b\left(\frac{\rho_1}{\rho_d}\right) \\ -(\eta^1_{4b} + \mathrm{i}\eta^2_{4b})\ \partial_b\left(\frac{\rho_{-1}}{\rho_d}\right) & -\eta^3_{4b}\ \partial_b\left(\frac{\rho_0}{\rho_d}\right) \end{pmatrix}.
\end{aligned} \tag{5.54}$$

But the fields A_a and Φ are not anti-hermitian and need to be transformed by the non-unitary gauge transformation (5.22). However, we find that

$$\begin{aligned}
&\psi_+(\lambda)\ \psi_-^\dagger(-\bar{\lambda}^{-1})|_{\lambda=0} \\
&= \frac{1}{\rho_0} \begin{pmatrix} \rho_0 + \frac{y^2}{b_1-b_2}(\rho_1^2 - \rho_0\rho_2) & -\frac{1}{\rho_d}(\rho_1^2 - \rho_0\rho_2)(\frac{y^2}{b_1-b_2}(\rho_1^2-\rho_0\rho_2) + 2\rho_0) \\ \frac{y^2}{b_1-b_2}\rho_d & \rho_0 + \frac{y^2}{b_1-b_2}(\rho_1^2 - \rho_0\rho_2) \end{pmatrix}
\end{aligned} \tag{5.55}$$

is not positive definite at some points. Furthermore, inserting (5.51) into (5.53), we find that this solution is singular on the x^3-axis. To suppose that ψ'_+ and ψ'_- are holomorphic in λ on the entire U_+ and U_-, respectively, is probably too restrictive as we will see in the next section. Nevertheless, we will show that this solution is related with the Dirac two-monopole solution by some singular gauge transformation.

5.6 Equivalence of Dirac and Wu-Yang Monopoles

As we have already mentioned, the Wu-Yang monopole is equivalent to the Dirac monopole solution embedded into $SU(2)$. There is no reason why it should not be valid for multi-monopole configurations.

Suppose that we have obtained a Wu-Yang n-monopole solutions via splitting of

$$f^{(n)}_{+-} = k_+ \begin{pmatrix} \rho^{(n)} & \lambda^{-n} \\ -\lambda^n & 0 \end{pmatrix} k_-, \tag{5.56}$$

where $\rho^{(n)} = \prod_{i=1}^n \frac{\lambda}{\eta_i}$ and $k^{(n)}_+$ and $k^{(n)}_-$ are some holomorphic functions on V_+ and V_-, respectively, such that $\det(f^{(n)}_{+-}) = 1$. Note that for n odd we have $k^{(n)}_\pm = \mathbf{1}_2$. We know that we will not change the configuration if we consider the transition function

$$\tilde f^{(n)}_{+-} = h^{(n)}_+ \; f^{(n)}_{+-} \; h^{(n)}_-, \tag{5.57}$$

where $h^{(n)}_+$ and $h^{(n)}_-$ are two arbitrary regular holomorphic matrix-valued functions on V_+ and V_-, respectively. Concretely, we take [30]

$$h^{(n)}_+ = \begin{pmatrix} \lambda^n(\rho^{(n)})^{-1} & 1 \\ 1 & 0 \end{pmatrix} (k^{(n)}_+)^{-1} \quad \text{and} \quad h^{(n)}_- = (k^{(n)}_-)^{-1} \begin{pmatrix} -\lambda^{-n}(\rho^{(n)})^{-1} & 1 \\ 1 & 0 \end{pmatrix}. \tag{5.58}$$

Note that $\lambda^n(\rho^{(n)})^{-1} = \prod_{i=1}^n \eta_i$ is holomorphic on V_+ and $\lambda^{-n}(\rho^{(n)})^{-1} = \prod_{i=1}^n \frac{\eta_i}{\lambda^2} = \prod_{i=1}^n \tilde\eta_i$ holomorphic on V_-. Inserting (5.58) in (5.57), we obtain

$$\tilde f^{(n)}_{+-} = \begin{pmatrix} (\rho^{(n)})^{-1} & 0 \\ 0 & \rho^{(n)} \end{pmatrix} = \begin{pmatrix} f^{(n)D}_{-+} & 0 \\ 0 & (f^{(n)D}_{+-})^{-1} \end{pmatrix}, \tag{5.59}$$

where $f^{(n)D}_{+-}$ gives via splitting a Dirac n monopole solution.

Two Wu-Yang monopoles. Here we come back to the construction of a Wu-Yang two-monopole solution. In the following we will use the notation from section 5.4. Suppose that $|\lambda_1|, |\lambda_2| \leq 1$ and restrict $f^{(2)}_{+-}$ to the circle which separates the points $\lambda = 0, \lambda_1, \lambda_2$ from the points $\lambda = -\bar\lambda_1^{-1}, -\bar\lambda_2^{-1}, \infty$. From the above equivalence we obtain some splitting of $f^{(2)}_{+-}$, which is different from the one we considered in section 5.5. Namely, using the results of (5.28) and (5.29), one finds that [3]

$$\psi^{NN}_+ = \underbrace{\frac{1}{\sqrt{\xi^{(2)}_N(\xi^{(2)}_N + \xi^{(2)}_S)}} \begin{pmatrix} \bar y_1 \bar y_2 & \xi^{(2)}_N \\ -\xi^{(2)}_N & y_1 y_2 \end{pmatrix}}_{g^{(2)}_N} \begin{pmatrix} \psi^N_+(\vec x_1)\, \psi^N_+(\vec x_2) & 0 \\ 0 & (\psi^N_+(\vec x_1)\, \psi^N_+(\vec x_2))^{-1} \end{pmatrix} h^{(2)}_+ \tag{5.60}$$

[3] I am grateful to A. D. Popov for suggesting this splitting.

is holomorphic in λ on $\{\lambda \in \mathbb{C}P^1 \mid |\lambda| < \min\{|\lambda_1|^{-1}, |\lambda_2|^{-1}\}\} \subset U_+$ and

$$\psi_-^{SS} = \underbrace{\frac{1}{\sqrt{\xi_S^{(2)}(\xi_N^{(2)}+\xi_S^{(2)})}} \begin{pmatrix} \xi_S^{(2)} & \bar{y}_1\bar{y}_2 \\ -y_1 y_2 & \xi_S^{(2)} \end{pmatrix}}_{g_S^{(2)}} \begin{pmatrix} \psi_-^S(\vec{x}_1)\ \psi_-^S(\vec{x}_2) & 0 \\ 0 & (\psi_-^S(\vec{x}_1)\ \psi_-^S(\vec{x}_2))^{-1} \end{pmatrix} (h_-^{(2)})^{-1} \tag{5.61}$$

in λ^{-1} on $\{\lambda \in \mathbb{C}P^1 \mid |\lambda| > \max\{|\lambda_1|, |\lambda_2|\}\} \subset U_-$. Here $\xi_N^{(2)} = \xi_{N1}\ \xi_{N2} = (r_1 + x_1^3)\ (r_2 + x_2^3)$ and $\xi_S^{(2)} = \xi_{S1}\ \xi_{S2} = (r_1 - x_1^3)\ (r_2 - x_2^3)$. Furthermore, (5.73) and (5.74) satisfy $f_{+-}^{(2)} = (\psi_+^{NN})^{-1}\ \psi_-^{SS}$ as well as the reality condition $(\psi_+^{NN}(-\bar{\lambda}^{-1}))^\dagger\ \psi_-^{SS}(\lambda) = \mathbf{1}_2$. Inserting (5.73) into (5.15) and (5.74) into (5.16) we obtain

$$\begin{aligned}
A_1 &= \frac{1}{r_1 r_2(\xi_N^{(2)}+\xi_S^{(2)})} \begin{pmatrix} -\mathrm{i}(x_1^2 r_2^2 + x_2^2 r_1^2) & x_1^3\bar{y}_2 r_2 + x_2^3\bar{y}_1 r_1 \\ -(x_1^3 y_2 r_2 + x_2^3 y_1 r_1) & \mathrm{i}(x_1^2 r_2^2 + x_2^2 r_1^2) \end{pmatrix} \\
A_2 &= \frac{-\mathrm{i}}{r_1 r_2(\xi_N^{(2)}+\xi_S^{(2)})} \begin{pmatrix} -(x_1^1 r_2^2 + x_2^1 r_1^2) & x_1^3\bar{y}_2 r_2 + x_2^3\bar{y}_1 r_1 \\ x_1^3 y_2 r_2 + x_2^3 y_1 r_1 & x_1^1 r_2^2 + x_2^1 r_1^2 \end{pmatrix} \\
A_3 &= \frac{r_1+r_2}{r_1 r_2(\xi_N^{(2)}+\xi_S^{(2)})} \begin{pmatrix} 0 & -\bar{y}_1\bar{y}_2 \\ y_1 y_2 & 0 \end{pmatrix} \\
\Phi &= \frac{-\mathrm{i}(r_1+r_2)}{r_1 r_2(\xi_N^{(2)}+\xi_S^{(2)})} \begin{pmatrix} r_1 x_2^3 + x_1^3 r_2 & \bar{y}_1\bar{y}_2 \\ y_1 y_2 & -(r_1 x_2^3 + x_1^3 r_2) \end{pmatrix}.
\end{aligned} \tag{5.62}$$

This solution is regular on $\mathbb{R}^3$ apart from the two points $r_1 = 0$ and $r_2 = 0$, as long as the monopoles are not aligned on the same line parallel to the x^3-axis. In this case $(\xi_N^{(2)}+\xi_S^{(2)}) = 0$ on the line connecting the two monopoles. But then one can consider another splitting [3] of $f_{+-}^{(2)}$:

$$\psi_+^{NS} = \underbrace{\frac{1}{\sqrt{\xi_{NS}^{(2)}(\xi_{NS}^{(2)}+\xi_{SN}^{(2)})}} \begin{pmatrix} \bar{y}_1 y_2 & \xi_{NS}^{(2)} \\ -\xi_{NS}^{(2)} & y_1\bar{y}_2 \end{pmatrix}}_{g_{NS}^{(2)}} \begin{pmatrix} \psi_+^N(\vec{x}_1)\ \psi_+^S(\vec{x}_2) & 0 \\ 0 & (\psi_+^N(\vec{x}_1)\ \psi_+^S(\vec{x}_2))^{-1} \end{pmatrix} h_+^{(2)} \tag{5.63}$$

with $\xi_{NS}^{(2)} = \xi_{N1}\ \xi_{S2} = (r_1 + x_1^3)\ (r_2 - x_2^3)$ and

$$\psi_-^{SN} = \underbrace{\frac{1}{\sqrt{\xi_{SN}^{(2)}(\xi_{NS}^{(2)}+\xi_{SN}^{(2)})}} \begin{pmatrix} \xi_{SN}^{(2)} & \bar{y}_1 y_2 \\ -y_1\bar{y}_2 & \xi_{SN}^{(2)} \end{pmatrix}}_{g_{SN}^{(2)}} \begin{pmatrix} \psi_-^S(\vec{x}_1)\ \psi_-^N(\vec{x}_2) & 0 \\ 0 & (\psi_-^S(\vec{x}_1)\ \psi_-^N(\vec{x}_2))^{-1} \end{pmatrix} (h_-^{(2)})^{-1} \tag{5.64}$$

with $\xi_{SN}^{(2)} = \xi_{S1}\ \xi_{S2} = (r_1 - x_1^3)\ (r_2 + x_2^3)$, which also satisfy $f_{+-}^{(2)} = (\psi_+^{NS})^{-1}\ \psi_-^{SN}$ as well as the reality condition $(\psi_+^{NS}(-\bar{\lambda}^{-1}))^\dagger\ \psi_-^{SN}(\lambda) = \mathbf{1}_2$. Again inserting (5.63) into (5.15) and

(5.64) into (5.16) we find

$$
\begin{aligned}
A_1' &= \frac{1}{r_1 r_2(\xi_{NS}^{(2)} + \xi_{SN}^{(2)})} \begin{pmatrix} \mathrm{i}(x_2^2 r_1^2 - x_1^2 r_2^2) & x_1^3 y_2 r_2 - x_2^3 \bar{y}_1 r_1 \\ -(x_1^3 \bar{y}_2 r_2 - x_2^3 y_1 r_1) & -\mathrm{i}(x_2^2 r_1^2 - x_1^2 r_2^2) \end{pmatrix} \\
A_2' &= \frac{-\mathrm{i}}{r_1 r_2(\xi_{NS}^{(2)} + \xi_{SN}^{(2)})} \begin{pmatrix} x_2^1 r_1^2 - x_1^1 r_2^2 & x_1^3 y_2 r_2 + x_2^3 \bar{y}_1 r_1 \\ x_1^3 \bar{y}_2 r_2 + x_2^3 y_1 r_1 & -(x_2^1 r_1^2 - x_1^1 r_2^2) \end{pmatrix} \\
A_3' &= \frac{r_1 + r_2}{r_1 r_2(\xi_{NS}^{(2)} + \xi_{SN}^{(2)})} \begin{pmatrix} 0 & -\bar{y}_1 y_2 \\ y_1 \bar{y}_2 & 0 \end{pmatrix} \\
\Phi' &= \frac{-\mathrm{i}(r_1 + r_2)}{r_1 r_2(\xi_{NS}^{(2)} + \xi_{SN}^{(2)})} \begin{pmatrix} x_1^3 r_2 - r_1 x_2^3 & \bar{y}_1 y_2 \\ y_1 \bar{y}_2 & -(x_1^3 r_2 - r_1 x_2^3) \end{pmatrix},
\end{aligned} \tag{5.65}
$$

where $(\xi_{NS}^{(2)} + \xi_{SN}^{(2)}) \neq 0$ on the line connecting the two monopoles. Summarize this: we probably need two gauge potentials and scalar fields defined on two patches covering the base space of the $SU(2)$ bundle over $\mathbb{R}^3 \setminus \{\vec{b}_1, \vec{b}_2\}$ to describe a Wu-Yang two-monopole configuration, when they sit on a line parallel to the x^3-axis. On the intersection of these two regions they are related by

$$
A' = f_{WY}^{(2)}\ A\ (f_{WY}^{(2)})^{-1} + f_{WY}^{(2)}\ \mathrm{d}(f_{WY}^{(2)})^{-1} \quad \text{and} \quad \Phi' = f_{WY}^{(2)}\ \Phi\ (f_{WY}^{(2)})^{-1} \tag{5.66}
$$

with the transition function $f_{WY}^{(2)} = \psi_+^{NS}\ (\psi_+^{NN})^{-1} = \psi_-^{SN}\ (\psi_-^{SS})^{-1}$ which does not depend on λ.

Note that the above construction is exactly equivalent to the embedding of the Dirac two-monopole solution (5.34) - (5.36) into $SU(2)$. One finds [4]

$$
(g_N^{(2)})^{-1}\ g_S^{(2)} = \begin{pmatrix} f_{NS}^{(2)D} & 0 \\ 0 & (f_{NS}^{(2)D})^{-1} \end{pmatrix} = f_{NS}^{(2)}. \tag{5.67}
$$

Furthermore, we have together with (5.32)

$$
\begin{aligned}
&A^{(2)N}\sigma_3 = f_{NS}^{(2)} A^{(2)S} \sigma_3 (f_{NS}^{(2)})^{-1} + f_{NS}^{(2)} \mathrm{d}(f_{NS}^{(2)})^{-1} \quad \text{and} \\
&\Phi^{(2)N}\sigma_3 = f_{NS}^{(2)} \Phi^{(2)S} \sigma_3 (f_{NS}^{(2)})^{-1}.
\end{aligned} \tag{5.68}
$$

Substituting (5.67) into (5.68), we obtain

$$
\begin{aligned}
&A = g_N^{(2)} A^{(2)N} \sigma_3 (g_N^{(2)})^{-1} + g_N^{(2)} \mathrm{d}(g_N^{(2)})^{-1} = g_S^{(2)} A^{(2)S} \sigma_3 (g_S^{(2)})^{-1} + g_S^{(2)} \mathrm{d}(g_S^{(2)})^{-1} \quad \text{and} \\
&\Phi = g_N^{(2)} \Phi^{(2)N} \sigma_3 (g_N^{(2)})^{-1} = g_S^{(2)} \Phi^{(2)S} \sigma_3 (g_S^{(2)})^{-1}
\end{aligned} \tag{5.69}
$$

where the components of (5.69) coincides with (5.62). Analogously, one finds that the components of $A' = g_{SN}^{(2)} A^{(2)M} \sigma_3 (g_{SN}^{(2)})^{-1} + g_{SN}^{(2)} \mathrm{d}(g_{SN}^{(2)})^{-1}$ and $\Phi' = g_{SN}^{(2)} \Phi^{(2)M} \sigma_3 (g_{SN}^{(2)})^{-1}$ coincides with (5.65). Conversely, for given A, A', $g_N^{(2)}$, $g_S^{(2)}$ and $g_{SN}^{(2)}$ one can obtain a gauge potential for the Dirac two-monopole solution

$$
\begin{aligned}
&A^{(2)N}\sigma_3 = (g_N^{(2)})^{-1} A g_N^{(2)} + (g_N^{(2)})^{-1} \mathrm{d} g_N^{(2)}, \quad A^{(2)S}\sigma_3 = (g_S^{(2)})^{-1} A g_S^{(2)} + (g_S^{(2)})^{-1} \mathrm{d} g_S^{(2)} \\
&A^{(2)M}\sigma_3 = (g_{SN}^{(2)})^{-1} A' g_{SN}^{(2)} + (g_{SN}^{(2)})^{-1} \mathrm{d} g_{SN}^{(2)}.
\end{aligned} \tag{5.70}
$$

[4] I would like to thank A. D. Popov for pointing this out before the publication of [38].

and similar for the scalar field.

Finally note that the solution from section 5.5 is related to the above solution by a singular gauge transforation. Recall that we have

$$f_{+-}^{(2)} = k_+(\psi'_+)^{-1}\ \psi'_- k_- = (\psi_+^{NN})^{-1}\ \psi_-^{SS} = (\psi_+^{NS})^{-1}\ \psi_-^{SN}. \tag{5.71}$$

Then the singular gauge transformation is given by

$$\psi_+^{NN} = g\ \psi'_+ k_+^{-1} \quad \text{and} \quad \psi_-^{SS} = g\ \psi'_- k_-, \tag{5.72}$$

where $g = \psi_+^{NN} k_+ (\psi'_+)^{-1} = \psi_-^{SS} k_-^{-1} (\psi'_-)^{-1}$ is independent of λ. Analogously, we can obtain ψ_+^{NS} and ψ_-^{SN}. Therefore, the configuration (5.53) is also equivalent to the Dirac two-monopole solution.

Wu-Yang multi-monopoles. In this way we will be able to construct a Wu-Yang n-monopole solution. It can be shown [38] that

$$\psi_- = \frac{1}{\sqrt{\xi_N^{(n)}(\xi_N^{(n)} + \xi_S^{(n)})}} \begin{pmatrix} \prod_{i=1}^n \bar{y}_i & \xi_N^{(n)} \\ -\xi_N^{(n)} & \prod_{i=1}^n y_i \end{pmatrix} \begin{pmatrix} \prod_{i=1}^n \psi_-^N(\vec{x}_i) & 0 \\ 0 & \prod_{i=1}^n (\psi_-^N(\vec{x}_i))^{-1} \end{pmatrix} h_-^{(n)} \tag{5.73}$$

with $\xi_N^{(n)} = \prod_{i=1}^n \xi_{Ni}$ and

$$\psi_+ = \frac{1}{\sqrt{\xi_S^{(n)}(\xi_N^{(n)} + \xi_S^{(n)})}} \begin{pmatrix} \xi_S^{(n)} & \prod_{i=1}^n \bar{y}_i \\ -\prod_{i=1}^n y_i & \xi_S^{(n)} \end{pmatrix} \begin{pmatrix} \prod_{i=1}^n \psi_+^S(\vec{x}_i) & 0 \\ 0 & \prod_{i=1}^n (\psi_+^S(\vec{x}_i))^{-1} \end{pmatrix} (h_+^{(n)})^{-1} \tag{5.74}$$

with $\xi_S^{(n)} = \prod_{i=1}^n \xi_{Si}$ satisfy $f_{+-}^{(n)} = (\psi_+)^{-1}\ \psi_-$ as well as reality condition $\psi_+^\dagger(-\bar{\lambda}^{-1})\ \psi_-(\lambda) = \mathbf{1}_2$. But as in the Dirac multi-monopole case we have to be concerned about solutions where some monopoles are aligned parallel to the x^3-axis. Then we probably need more coverings of $\mathbb{R}^3 \setminus \{\vec{b}_1, \dots \vec{b}_n\}$ for a global description of the multi-monopole solution.

6 Monopoles in Noncommutative Coordinates

6.1 Monopoles in String Theory

Magnetic monopoles are very important in string theory since they should exist if string theory is correct [54]. They appear in a D1-D3-brane system (see e.g. [20, 16, 21]). Two Dirac monopoles can be interpreted as 2 D1-branes each with one end attached to a D3-brane. The end of a D1-brane has magnetic charge on the D3-brane, so if a D1-brane touches a D3-brane this creates a magnetic monopole. The same situation but with an additional magnetic background field on the D3-brane has to be considered in noncommutative coordinates.

6.2 Noncommutative $\mathbb{R}^3_\theta$

We will use the operator formulation (cf. section 4.2) in order to obtain the noncommutative deformation $\mathbb{R}^3_\theta$ of $\mathbb{R}^3$. We regard the coordinates in $\mathbb{R}^3$ as operators $\hat{x}^1$, $\hat{x}^2$ and $\hat{x}^3$, which obey the following commutation relation:

$$[\hat{x}^a, \hat{x}^b] = \mathrm{i}\ \theta^{ab} \tag{6.1}$$

with a constant antisymmetric tensor θ^{ab}. The coordinates can be chosen in such a way that the only non-vanishing entries of the matrix (θ^{ab}) are

$$\theta^{12} = -\theta^{21} = \theta \geq 0. \tag{6.2}$$

Hence, the coordinate $\hat{x}^3$ is commutative and we will keep on denoting it by x^3. As in section 4.2, we introduce the complex coordinate $\hat{y} = \hat{x}^1 + \mathrm{i}\hat{x}^2$. Thus, the commutation relation reads

$$[\hat{y}, \hat{\bar{y}}] = 2\theta. \tag{6.3}$$

Now one regards $\hat{\bar{y}}$ and $\hat{y}$ as creation and annihilation operators respectively, which act on a one-oscillator Fock space $\mathcal{H}$ with the basis

$$|n_1\rangle = \frac{1}{\sqrt{n_1!\ (2\theta)^n}}\ (\hat{\bar{y}})^{n_1}\ |0\rangle \qquad \text{for } n_1 \in \mathbb{N}. \tag{6.4}$$

The operator formulation turns any function f on $\mathbb{R}^3$ into operators $\hat{f}$ acting on this Fock space (6.4). The two spatial derivatives are described by the commutators

$$\hat{\partial}_y\ \hat{f} = -\frac{1}{2\theta}\ [\hat{\bar{y}}, \hat{f}] \qquad \text{and} \qquad \hat{\partial}_{\bar{y}}\ \hat{f} = \frac{1}{2\theta}\ [\hat{y}, \hat{f}] \tag{6.5}$$

and the integration over d^3x by the following trace over the Fock space:

$$2\pi\theta \int_{\mathbb{R}} \mathrm{d}x^3\ \mathrm{tr}(\hat{f}). \tag{6.6}$$

6.3 Wu-Yang Monopoles in Noncommutative Coordinates

Having replaced y and $\bar{y}$ by the operators $\hat{y}$ and $\hat{\bar{y}}$ respectively, the equations in section 5.2 and 5.3 retain their form, but every function is now operator-valued. For a more detailed discussion in terms of deformed minitwistor space see e.g. [30]. As in section 5.5, we will recall the results for a Wu-Yang one-monopole solution first before we proceed to the two-monopole case. Irrespective of the problem with the anti-hermiticity of the gauge potential, which occurred in commutative coordinates, we will outline the calculations in noncommutative coordinates.

Let us gather some general results. We have

$$\hat{\eta} = \hat{y} - 2\lambda x^3 - \lambda^2 \hat{\bar{y}} = (\lambda + \hat{y}\hat{\xi}^{-2})\ (\hat{\xi}^2 - \lambda\hat{\bar{y}}), \tag{6.7}$$

where

$$\hat{\xi} = (2\theta)^{\frac{1}{4}} \sum_{n_1 \geq 0} \xi_{n_1}(x^3)|n_1\rangle\langle n_1| \tag{6.8}$$

with ξ_{n_1} being a function of x^3, chosen in a way that $\hat{\xi}$ solves the equation

$$\hat{\xi}^2 - \hat{y}\ \hat{\xi}^{-2}\ \hat{\bar{y}} = -2x^3. \tag{6.9}$$

These coefficients ξ_{n_1} are found in [20]

$$\xi_{n_1}(x^3) = \sqrt{\frac{\Upsilon_{n_1-1}(\frac{x^3}{\sqrt{2\theta}})}{\Upsilon_{n_1}(\frac{x^3}{\sqrt{2\theta}})}}\ , \tag{6.10}$$

where $\Upsilon_k(x)$ are given by

$$\Upsilon_{-1} = 1 \qquad \text{and} \qquad \Upsilon_k(x) = \int_0^\infty \mathrm{d}p\ \frac{p^k}{k!}\ \exp\big(2xp - \frac{1}{2}p^2\big) \qquad \text{for} \quad k \geq 0. \tag{6.11}$$

Furthermore, we can calculate the inverse of any matrix $\hat{m} = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix}$ in noncommutative coordinates, supposing that $\hat{a}$ and $\hat{d} - \hat{c}\hat{a}^{-1}\hat{b}$ are invertible, by

$$\hat{m}^{-1} = \begin{pmatrix} \hat{a}^{-1} + \hat{a}^{-1}\hat{b}(\hat{d} - \hat{c}\hat{a}^{-1}\hat{b})^{-1}\hat{c}\hat{a}^{-1} & -\hat{a}^{-1}\hat{b}(\hat{d} - \hat{c}\hat{a}^{-1}\hat{b})^{-1} \\ -(\hat{d} - \hat{c}\hat{a}^{-1}\hat{b})^{-1}\hat{c}\hat{a}^{-1} & (\hat{d} - \hat{c}\hat{a}^{-1}\hat{b})^{-1} \end{pmatrix}. \tag{6.12}$$

It is also useful to recall that for every operator $\hat{a}$ we have

$$(1 - \hat{a})^{-1} = \sum_{k=0}^{\infty}\ \hat{a}^k. \tag{6.13}$$

One Wu-Yang monopole in noncommutative coordinates. We find for the components of the gauge potential and the scalar field of the Wu-Yang monopole solution in noncommutative coordinates [30]

$$\begin{aligned} \hat{A}_a &= \epsilon_{abc}\frac{\sigma_c}{2\mathrm{i}}(\hat{\rho}_0^{\frac{1}{2}}\hat{\partial}_b\hat{\rho}_0^{-\frac{1}{2}} - \hat{\rho}_0^{-\frac{1}{2}}\hat{\partial}_b\hat{\rho}_0^{\frac{1}{2}}) + \frac{1}{2}(\hat{\rho}_0^{-\frac{1}{2}}\hat{\partial}_b\hat{\rho}_0^{\frac{1}{2}} + \hat{\rho}_0^{\frac{1}{2}}\hat{\partial}_b\hat{\rho}_0^{-\frac{1}{2}})\mathbf{1}_2, \\ \hat{\Phi} &= \frac{\sigma_a}{2\mathrm{i}}(\hat{\rho}_0^{\frac{1}{2}}\hat{\partial}_a\hat{\rho}_0^{-\frac{1}{2}} - \hat{\rho}_0^{-\frac{1}{2}}\hat{\partial}_a\hat{\rho}_0^{\frac{1}{2}}), \end{aligned} \tag{6.14}$$

where $\hat\rho_0 = (2\hat R)^{-1}$ with $2\hat R = (1 + \sum_{n=0}^{\infty}[\hat y\hat\xi^{-2}, (\hat{\bar y}\hat\xi^{-2})^{n+1}]\ (-\hat y\hat\xi^{-2})^n)^{-1}\ 2(\hat\xi^2 + x^3)$.

Two Wu-Yang monopoles in noncommutative coordinates. The holomorphic transition function in noncommutative coordinates $\hat f^{(2)}_{+-}$ takes the same form as in (5.41), but we replace the function $\rho^{(2)}(x^a,\lambda)$ by an operator-valued function $\hat\rho^{(2)}(\hat x^a,\lambda)$. In distinction to section 5.5 the condition (5.40) now reads

$$(\hat\rho^{(2)}(\hat x^a, -\bar\lambda^{-1}))^\dagger = \hat\rho^{(2)}(\hat x^a,\lambda) \tag{6.15}$$

and we need to restrict $\hat f^{(2)}_{+-}$ to the circle $S^1 = \{\lambda \in \mathbb{C}P^1 \mid |\lambda| = 1\}$, on which $\hat\eta_i$, a squeezed operator [36], is invertible. The operator-valued version of (5.48) still satisfies the splitting equation

$$\hat\psi_+ \hat f^{(2)}_{+-} = \hat\psi_-, \tag{6.16}$$

but we have to calculate the inverse of $\hat\psi_+$ and the one of $\hat\psi_-$ via (6.12).

As in section 5.5, we specialize to

$$\hat\rho^{(2)} = 2\ \lambda^2\ (b_1 - b_2)\ (\hat\eta_1\ \hat\eta_2)^{-1} = \lambda\ (\hat\eta_1^{-1} - \hat\eta_2^{-1}), \tag{6.17}$$

where we assume that the monopoles sit on the x^3-axis. One could try to evaluate the integrals

$$\begin{aligned}\hat\rho_0 &= \oint_{|\lambda|=1} \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}\lambda}\ \hat\rho^{(2)} = \oint_{|\lambda|=1} \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}\ (\hat\eta_1^{-1} - \hat\eta_2^{-1}) \qquad \text{and}\\ \hat\rho_1 &= \oint_{|\lambda|=1} \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}\lambda^2}\ \hat\rho^{(2)} = 2(b_1 - b_2)\ \oint_{|\lambda|=1} \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}\ \hat\eta_2^{-1}\ \hat\eta_1^{-1}\\ &= -\hat\rho^\dagger_{-1},\end{aligned} \tag{6.18}$$

but we can even read the coefficients $\hat\rho_0$, $\hat\rho_1$ and $\hat\rho_{-1}$ directly from the Laurent series. We introduce $2\hat{\hat R}_i = 2(\hat\xi_i^2 + x_i^3)$ with $\hat\xi_i = \hat\xi(x_i^3)$ and calculate, using (6.9),

$$2\hat{\hat R}_i = \hat\xi_i^2 + \hat y\hat\xi_i^{-2}\hat{\bar y} = (\lambda + \hat y\hat\xi_i^{-2})\hat{\bar y} + (\hat\xi_i^2 - \lambda\hat{\bar y}). \tag{6.19}$$

Then we find

$$\begin{aligned}\lambda\hat\eta_i^{-1} &= (2\hat{\hat R}_i)^{-1}\ \Big(\lambda\ (\lambda + \hat y\hat\xi_i^{-2})^{-1} + \lambda\ (\lambda + \hat y\hat\xi_i^{-2})\ \hat{\bar y}\ (\hat\xi_i^2 - \lambda\hat{\bar y})^{-1}\ (\lambda + \hat y\hat\xi_i^{-2})^{-1}\Big)\\ &= (2\hat{\hat R}_i)^{-1}\ \left(\sum_{k=0}^{\infty}(-\frac{\hat y\hat\xi_i^{-2}}{\lambda})^k + (\lambda + \hat y\hat\xi_i^{-2})\ \sum_{k=0}^{\infty}(\hat{\bar y}\hat\xi_i^{-2}\ \lambda)^{k+1}\ (\lambda + \hat y\hat\xi_i^{-2})^{-1}\right)\\ &= (2\hat{\hat R}_i)^{-1}\ \left(\sum_{k=0}^{\infty}(-\frac{\hat y\hat\xi_i^{-2}}{\lambda})^k + \sum_{k=0}^{\infty}(\hat{\bar y}\hat\xi_i^{-2}\ \lambda)^{k+1} + \sum_{k,m=0}^{\infty}[\hat y\hat\xi_i^{-2}, (\hat{\bar y}\hat\xi_i^{-2})^{k+1}]\ (-\hat y\hat\xi_i^{-2})^m\ \lambda^{k-m}\right).\end{aligned} \tag{6.20}$$

We redefine $2\hat{\hat R}_i$ by $2\hat R_i = \Big(1 + \sum_{k=0}^{\infty}[\hat y\hat\xi_i^{-2}, (\hat{\bar y}\hat\xi_i^{-2})^{k+1}]\ (-\hat y\hat\xi_i^{-2})^k\Big)^{-1}\ 2\hat{\hat R}_i$ and finally obtain

$$\hat\rho_0 = (2\hat R_1)^{-1} - (2\hat R_2)^{-1} \quad \text{and} \quad \hat\rho_{-1} = -(2\hat R_1)^{-1}\hat y\hat\xi_1^{-2} + (2\hat R_2)^{-1}\hat y\hat\xi_2^{-2} = -\hat\rho_1^\dagger. \tag{6.21}$$

One can easily see that $\hat{\rho}_0 = (2\hat{R}_1)^{-1}(1 - (2\hat{R}_1)(2\hat{R}_2)^{-1})$ is invertible. Note that in the commutative limit we have $2\hat{R}_i \to 2r_i$ and $\hat{\xi}_i^2 = \hat{\tilde{R}}_i - x_i^3 \to r_i - x_i^3$. Thus, (6.21) coincides with (5.51).

Let us suppose that $\hat{k}_+|_{\lambda=0}$ depends on y only and $\hat{k}_-|_{\lambda=\infty}$ depends on $\bar{y}$ only. Then we choose $\hat{a}_0 = 1$, $\hat{a}_1 = -\hat{\rho}_1\ \hat{\rho}_0^{-1}$, $\hat{a}_2 = 0$, $\hat{c}_0 = 0$, $\hat{c}_1 = -\hat{\rho}_{-1}\ \hat{\rho}_0^{-1}$ and $\hat{c}_2 = 1$ and we get, using the abbreviations $\hat{\rho}_{1,1} = \hat{\rho}_1\ \hat{\rho}_0^{-1}\ \hat{\rho}_1$, $\hat{\rho}_{1,-1} = \hat{\rho}_1\ \hat{\rho}_0^{-1}\ \hat{\rho}_{-1}$, $\hat{\rho}_{-1,1} = \hat{\rho}_{-1}\ \hat{\rho}_0^{-1}\ \hat{\rho}_1$ and $\hat{\rho}_{-1,-1} = \hat{\rho}_{-1}\ \hat{\rho}_0^{-1}\ \hat{\rho}_{-1}$,

$$\begin{aligned} \hat{\psi}_+|_{\lambda=0} &= \begin{pmatrix} 1 & \hat{\rho}_2 - \hat{\rho}_{1,1} \\ 0 & \hat{\rho}_0 - \rho_{-1,1} \end{pmatrix}, \qquad \hat{\psi}_+^{-1}|_{\lambda=0} = \begin{pmatrix} 1 & -(\hat{\rho}_2 - \hat{\rho}_{1,1})(1 - \hat{\rho}_0^{-1}\hat{\rho}_{-1,1})^{-1}\hat{\rho}_0^{-1} \\ 0 & (1 - \hat{\rho}_0^{-1}\hat{\rho}_{-1,1})^{-1}\hat{\rho}_0^{-1} \end{pmatrix} \\ \hat{\psi}_-|_{\lambda=\infty} &= \begin{pmatrix} \hat{\rho}_0 - \hat{\rho}_{1,-1} & 0 \\ \hat{\rho}_{-2} - \hat{\rho}_{-1,-1} & 1 \end{pmatrix}, \qquad \hat{\psi}_-^{-1}|_{\lambda=\infty} = \begin{pmatrix} (1 - \hat{\rho}_0^{-1}\hat{\rho}_{1,-1})^{-1}\hat{\rho}_0^{-1} & 0 \\ -(\hat{\rho}_{-2} - \hat{\rho}_{-1,-1})(1 - \hat{\rho}_0^{-1}\hat{\rho}_{1,-1})^{-1}\hat{\rho}_0^{-1} & 1 \end{pmatrix}. \end{aligned} \tag{6.22}$$

Inserting (6.22) into the noncommutative version of (5.49) and using (5.45), we obtain

$$\begin{aligned} \hat{A}_1 &= \begin{pmatrix} (\hat{\rho}_0 - \hat{\rho}_{1,-1})(\hat{\partial}_y(\hat{\rho}_0 - \hat{\rho}_{1,-1})^{-1}) & (\hat{\partial}_{\bar{y}}\hat{\rho}_{1,1} - \frac{1}{2}\partial_3\hat{\rho}_1)(\hat{\rho}_0 - \hat{\rho}_{-1,1})^{-1} \\ (\hat{\partial}_y\hat{\rho}_{-1,-1} + \frac{1}{2}\partial_3\hat{\rho}_{-1})(\hat{\rho}_0 - \hat{\rho}_{1,-1})^{-1} & (\hat{\rho}_0 - \hat{\rho}_{-1,1})(\hat{\partial}_{\bar{y}}(\hat{\rho}_0 - \hat{\rho}_{-1,1})^{-1}) \end{pmatrix}, \\ \hat{A}_2 &= -\mathrm{i}\begin{pmatrix} -(\hat{\rho}_0 - \hat{\rho}_{1,-1})(\hat{\partial}_y(\hat{\rho}_0 - \hat{\rho}_{1,-1})^{-1}) & (\hat{\partial}_{\bar{y}}\hat{\rho}_{1,1} - \frac{1}{2}\partial_3\hat{\rho}_1)(\hat{\rho}_0 - \hat{\rho}_{-1,1})^{-1} \\ -(\hat{\partial}_y\hat{\rho}_{-1,-1} + \frac{1}{2}\partial_3\hat{\rho}_{-1})(\hat{\rho}_0 - \hat{\rho}_{1,-1})^{-1} & (\hat{\rho}_0 - \hat{\rho}_{-1,1})(\hat{\partial}_{\bar{y}}(\hat{\rho}_0 - \hat{\rho}_{-1,1})^{-1}) \end{pmatrix}, \\ \hat{A}_3 &= \begin{pmatrix} (\hat{\rho}_0 - \hat{\rho}_{1,-1})(\partial_3(\hat{\rho}_0 - \hat{\rho}_{1,-1})^{-1}) & (\partial_3\hat{\rho}_{1,1} + 2\hat{\partial}_y\hat{\rho}_1)(\hat{\rho}_0 - \hat{\rho}_{-1,1})^{-1} \\ (\partial_3\hat{\rho}_{-1,-1} - 2\hat{\partial}_{\bar{y}}\hat{\rho}_{-1})(\hat{\rho}_0 - \hat{\rho}_{1,-1})^{-1} & (\hat{\rho}_0 - \hat{\rho}_{-1,1})(\partial_3(\hat{\rho}_0 - \hat{\rho}_{-1,1})^{-1}) \end{pmatrix}, \\ \hat{\Phi} &= -\mathrm{i}\begin{pmatrix} (\hat{\rho}_0 - \hat{\rho}_{1,-1})(\partial_3(\hat{\rho}_0 - \hat{\rho}_{1,-1})^{-1}) & -(\partial_3\hat{\rho}_{1,1} + 2\hat{\partial}_y\hat{\rho}_1)(\hat{\rho}_0 - \hat{\rho}_{-1,1})^{-1} \\ (\partial_3\hat{\rho}_{-1,-1} - 2\hat{\partial}_{\bar{y}}\hat{\rho}_{-1}))(\hat{\rho}_0 - \hat{\rho}_{1,-1})^{-1} & -(\hat{\rho}_0 - \hat{\rho}_{-1,1})(\partial_3(\hat{\rho}_0 - \hat{\rho}_{-1,1})^{-1}) \end{pmatrix}. \end{aligned} \tag{6.23}$$

In the commutative limit ($\theta \to 0$) this solution coincides with (5.53), which is equivalent to the Wu-Yang two-monopole solution. We have to find a noncommutative equivalent of the nonsingular gauge transformation (5.72). However, this is technical not easy and we will not try to do it here.

6.4 Dirac Monopoles in Noncommutative Coordinates

We will show how one can obtain a splitting for the Dirac multi-monopole configuration in noncommutative coordinates from the Wu-Yang multi-monopole solution calculated above. Equations (5.57) - (5.59) retain their form in noncommutative coordinates. We have to split

$$\begin{pmatrix} \hat{f}_{-+}^{(n)D} & 0 \\ 0 & (\hat{f}_{+-}^{(n)D})^{-1} \end{pmatrix} = (\hat{\psi}_+^{\mathrm{diag}})^{-1}\ \hat{\psi}_-^{\mathrm{diag}} \tag{6.24}$$

to obtain a Dirac two-monopole solution in noncommutative coordinates. But this work is almost done. We need to find two matrices $\hat{w}_+$ and $\hat{w}_-$ holomorphic on U_+ and U_- respectively such that $\hat{w}_+\hat{\psi}_+\hat{h}_+^{-1}$ and $\hat{w}_-\hat{\psi}_-\hat{h}_-$ are diagonal. Furthermore, $\hat{w}_+$ and $\hat{w}_-$ need to be invertible and equal to each other on $S^1 = \{\lambda \in \mathbb{C}P^1 \mid |\lambda| = 1\}$. We take

$$\hat{w}_+ = \begin{pmatrix} -((\hat{\psi}_+)_{21} - (\hat{\psi}_+)_{22}\lambda^n(\hat{\rho}^{(n)})^{-1})((\hat{\psi}_+)_{11} - (\hat{\psi}_+)_{12}\lambda^n(\hat{\rho}^{(n)})^{-1})^{-1} & 1 \\ 1 & -(\hat{\psi}_+)_{12}(\hat{\psi}_+)_{22}^{-1} \end{pmatrix},$$

$$\hat{w}_- = \begin{pmatrix} -(\hat{\psi}_-)_{21}(\hat{\psi}_-)_{11}^{-1} & 1 \\ 1 & -((\hat{\psi}_-)_{12} - (\hat{\psi}_-)_{11}\lambda^{-n}(\hat{\rho}^{(n)})^{-1})((\hat{\psi}_-)_{22} - (\hat{\psi}_-)_{21}\lambda^{-n}(\hat{\rho}^{(n)})^{-1})^{-1} \end{pmatrix} \tag{6.25}$$

and find for the (11)-element of (6.24)

$$\begin{aligned} \hat{f}_{+-}^{(n)D} = {} & ((\hat{\psi}_+)_{11} - (\hat{\psi}_+)_{12}(\hat{\psi}_+)_{22}^{-1}(\hat{\psi}_+)_{21})^{-1} \\ & ((\hat{\psi}_-)_{11} - ((\hat{\psi}_-)_{12} - (\hat{\psi}_-)_{11}\lambda^{-n}(\hat{\rho}^{(n)})^{-1})((\hat{\psi}_-)_{22} - (\hat{\psi}_-)_{21}\lambda^{-n}(\hat{\rho}^{(n)})^{-1})^{-1}(\hat{\psi}_-)_{21}) \end{aligned} \tag{6.26}$$

Dirac monopole in noncommutative coordinates. For the case $n = 1$ (6.26) becomes

$$\hat{\rho}^{(1)} = (\hat{\rho}_0 + \hat{\rho}_+)\hat{\rho}_0^{-\frac{1}{2}}\ \hat{\rho}_0^{\frac{1}{2}}(1 + \lambda^{-1}(\hat{\rho}^{(1)})^{-1}(1 - \lambda\hat{\rho}_-\ \lambda^{-1}(\hat{\rho}^{(1)})^{-1})^{-1}\lambda\hat{\rho}_-). \tag{6.27}$$

But there is a much simpler splitting [30]

$$\hat{\rho}^{(1)} = (\hat{\xi} - \lambda\hat{\xi}^{-1}\hat{\bar{y}})^{-1}\ (\hat{\xi} + \lambda^{-1}\hat{y}\hat{\xi}^{-1})^{-1}. \tag{6.28}$$

Then the Dirac monopole in noncommutative coordinates is given by

$$\hat{A}_1 = -\frac{1}{\theta}[\hat{x}_1, \hat{\xi}]\hat{\xi}^{-1}, \qquad \hat{A}_2 = \frac{1}{\theta}[\hat{x}_2, \hat{\xi}]\hat{\xi}^{-1}, \qquad \hat{A}_3 = 0, \qquad \hat{\Phi} = -\mathrm{i}(\partial_3\hat{\xi})\hat{\xi}^{-1}. \tag{6.29}$$

Two Dirac monopoles in noncommutative coordinates. For $n = 2$ one can also obtain a splitting via (6.26) but it becomes very complicated and we do not want to consider it here.

7 Conclusion

We have seen that the construction of instanton and monopole solutions is a vast and fascinating subject. Furthermore, there are still a lot of unresolved questions and problems. In this diploma thesis we wanted to give a survey of instantons and monopoles and we have looked at some of the problems occurring in their construction.

In commutative coordinates, we have investigated a special ansatz for a multi-instanton solution in the ADHM construction and have shown that this ansatz produces regular solutions at finite points. Then, in noncommutative coordinates, we have depicted two different ways of how to obtain a noncommutative generalization of the 't Hooft instantons. The first one was a transformation of the commutative 't Hooft ansatz to some noncommutative version. In the second approach we have made the moduli space noncommutative, but in a very specific way. This has produced instanton-like solutions, which are also interesting in their own right. We succeeded in generalizing the construction of instanton solutions, which are regular at finite points, for a single instanton. Then we have calculated the case of two instantons up to normalization.

In the second part of this diploma thesis, we have described Dirac multi-monopole solutions obtained by some splitting of a holomorphic function. We have pointed out that we may need more than two coverings of $\mathbb{R}^3 \setminus \{\vec{b}_1, \dots \vec{b}_n\}$ for a global description of the gauge potential if two or more monopoles are aligned parallel to the x^3-axis. In our attempt to construct a Wu-Yang two-monopole solution, we have encountered some problems, which occur if one considers a splitting of the holomorphic transition function, where one part of this splitting is holomorphic on the entire U_+ and the other part is holomorphic on the entire U_-. Finally, we have obtained a Wu-Yang two-monopole solution and described the equivalence of the Dirac and the Wu-Yang monopoles. Here we have again found that we may need two coverings of $\mathbb{R}^3 \setminus \{\vec{b}_1, \vec{b}_2\}$ for a global description of the gauge potential if the two monopoles are aligned parallel to the x^3-axis. In the end, we have discussed some attempts of how to generalize the construction of a Wu-Yang two-monopole solution to noncommutative coordinates.

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Selbständigkeitserklärung

Hiermit erkläre ich, die Diplomarbeit selbständig verfasst und keine anderen als die angegebenen Hilfsmittel verwendet zu haben.

Hannover, 28.10.2004