# Dilogarithm Identities and Characters of Exceptional Rational Conformal Field Theories

Diplomarbeit

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## ABSTRACT

Based on our recently published article *Fermionic Expressions for the Characters of* $c_{p,1}$ *Logarithmic Conformal Field Theories* in the journal Nuclear Physics B, fermionic sum representations for the characters of the $\mathcal{W}(2, 2p-1, 2p-1, 2p-1)$ series of triplet algebras are obtained, providing further evidence that these logarithmic theories constitute well-defined rational conformal field theories. Furthermore, after an investigation of Nahm's conjecture, fermionic character expressions for other conformal field theories such as the minimal Virasoro models and $SU(2)$ Wess-Zumino-Witten (WZW) models are given, some of the latter also being new. In combination with their known bosonic counterparts, fermionic character expressions give rise to so-called bosonic-fermionic $q$-series identities, closely related to the famous Rogers-Ramanujan and Andrews-Gordon identities. Additionally, it is displayed how fermionic character expressions imply dilogarithm identities for the effective central charge of the conformal field theory in question. In the case of the triplet algebras, this results in an infinite series of dilogarithm identities. Since a proof for this series of identities already exists, this strongly supports the corresponding fermionic character expressions. In general, fermionic sum representations for characters give rise to an interpretation of the corresponding theory in terms of quasi-particles which obey generalized exclusion statistics. This quasi-particle content is discussed for conformal field theories which admit fermionic character expressions.

## ZUSAMMENFASSUNG

Basierend auf unserem Artikel *Fermionic Expressions for the Characters of* $c_{p,1}$ *Logarithmic Conformal Field Theories*, veröffentlicht im Journal Nuclear Physics B, werden in dieser Arbeit fermionische Summendarstellungen der Charaktere der $\mathcal{W}(2, 2p-1, 2p-1, 2p-1)$-Serie von Triplettalgebren gewonnen. Die Existenz dieser Darstellungen liefert weitere Indizien, daß es sich bei diesen logarithmischen Theorien um wohldefinierte rationale konforme Feldtheorien handelt. Darüberhinaus werden nach einer Beschäftigung mit der Nahm-Vermutung fermionische Charakterausdrücke für andere konforme Feldtheorien wie die minimalen Virasoro-Modelle und die $SU(2)$-Wess-Zumino-Witten-Modelle angegeben, unter den letzteren auch bisher unbekannte. In Kombination mit ihren bekannten bosonischen Gegenstücken implizieren diese fermionischen Charakterdarstellungen sogenannte Bose-Fermi-$q$-Reihen-Identitäten, welche in engem Zusammenhang zu den berühmten Rogers-Ramanujan- und Andrews-Gordon-Identitäten stehen. Desweiteren wird gezeigt, wie fermionische Charakterdarstellungen zu dilogarithmischen Identitäten für die effektiven zentralen Ladungen der betrachteten konformen Feldtheorie führen. Für unsere Ergebnisse bezüglich der Triplettalgebren resultiert dies in einer unendlichen Reihe solcher Identitäten. Das Vorhandensein eines Beweises unterstützt die entsprechenden fermionischen Charakterausdrücke. Letztere ermöglichen eine Interpretation des zugrundeliegenden Systems anhand von fermionischen Quasiteilchen, welche verallgemeinerten Ausschlußprinzipien genügen. Der Quasiteilcheninhalt für konforme Feldtheorien, die fermionische Charakterausdrücke zulassen, wird diskutiert.

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# Introduction and Overview

The concepts of bosons and fermions may be defined through their spectra or in terms of commutation and anti-commutation relations. Due to the spin-statistics theorem, both approaches are equivalent. In three space dimensions, bosons and fermions are quite distinct. However, in $1+1$ dimensions, i.e. one space dimension and one time dimension, the difference between fermionic and bosonic particles is merely of a statistical nature. This is called Bose-Fermi correspondence in $1 + 1$ dimensions.

Meanwhile, conformally invariant quantum field theories in two dimensions are an interesting object of study. There are mainly two areas in theoretical physics where they are commonly used. These are on the one hand the perturbative vacua of string theory, where the world-sheet of a string is described by a Riemann surface, i.e. a two-dimensional manifold described in terms of complex coordinates together with a conformal structure. On the other hand, conformal field theories serve as statistical mechanics models for second order phase transitions at the critical point. Indeed, at the phase transition, typical configurations have fluctuations on all length scales. Thus, a field theory describing the model at the critical point should at least be invariant under scale transformations.[1]

The main reason why conformal field theory in two dimensions is so useful is that in this case, conformal transformations coincide with analytic transformations on the complex plane. Because an analytic transformation is always expressible as a Laurent series, an infinite number of parameters is necessary to specify a given infinitesimal conformal transformation, resulting in an infinite-dimensional conformal algebra which consists of two commuting copies of the Virasoro algebra, a holomorphic and an antiholomorphic one. A conformal field theory is defined to be invariant under these transformations. Hence, the Virasoro algebra constitutes a symmetry algebra of the theory.

Symmetry is a fundamental principle of physics. Symmetries of a physical problem lead to conserved quantities. In quantum field theory, particles are defined as finite-dimensional irreducible representations of the space-time (and internal) symmetry groups. In statistical mechanics, symmetry is employed in characterizing degrees of freedom and types of interaction. The more symmetries there are in a physical problem, the more likely it is that the problem is exactly solvable, i.e. without the use of perturbation theory. However, since quantum field theories have infinitely many degrees of freedom, they are in almost any cases not exactly solvable. In general, quantum field theory only features Poincaré invariance, i.e. invariance under translations of time and space, rotations and Lorentz boosts. But if the considered theory is massless, it is furthermore invariant under scale transformations and, more generally, in almost any cases also under conformal transformations. In $d > 2$ dimensions with signature[2] $(p, q)$, the conformal group is isomorphic to the group $SO(p+1, q+1)$

[1] For an introduction to these two areas, the reader is referred to e.g. [GSW87] and [Car89], respectively.
[2] The metric is of the form $(\eta_{\mu\nu}) = \mathrm{diag}(\underbrace{-1, -1, \ldots, -1}_{q \text{ times}}, \underbrace{+1, +1, \ldots, +1}_{p \text{ times}})$.

of special pseudo-orthogonal matrices in $d$ dimensions and hence finite-dimensional. But since in two dimensions the conformal algebra becomes infinite-dimensional and since it is also possible to directly define the stress-energy tensor as a quantum field, which preserves the conformal symmetry while quantizing, certain theories may in principle be computed exactly.

Quantizing the conformal field theory, a conformal anomaly $c$ enters the game. It is called central charge and parameterizes the theory. For example, there is the series of models with central charge $c_{p,p'} = 1 - 6\frac{(p-p')^2}{pp'}$ with $p, p' \in \mathbb{Z}_{\geq 2}$ integer and coprime. It was shown by Belavin, Polyakov and Zamolodchikov [BPZ84] that the space of states of these theories decomposes into only a finite number of irreducible representations of the left and right Virasoro algebra, corresponding to highest weights $h_{r,s}^{p,p'} = \frac{(pr-p's)^2-(p-p')^2}{4pp'}$, $1 \leq r \leq p'-1$ and $1 \leq s \leq p-1$. Hence, these have been termed *minimal models*, denoted by $\mathcal{M}(p,p')$.

The zero mode $L_0$ of the Virasoro algebra admits a natural gradation of the representation space into subspaces of fixed $L_0$ eigenvalue. The character of a representation displays the number of linearly independent states in each of these subspaces as a series expansion in some formal variable $q$, where the dimension $k$ of the subspace with fixed $L_0$ eigenvalue $n$ is indicated by a summand $kq^n$ in this series. If the character of a representation of the Virasoro algebra is defined as $\chi_h(q) = \mathrm{Tr} q^{L_0 - \frac{c}{24}}$, then the minimal model partition function can be written as a finite sum of products of holomorphic and antiholomorphic characters. Belavin, Polyakov and Zamolodchikov showed that because of the infinite-dimensional symmetry, all correlation functions of primary fields can be computed exactly in these cases. In general, if the Hilbert space of states of a conformal field theory decomposes into a finite sum of irreducible representations, then the theory is said to be *rational*.

A conformal field theory with periodic boundary conditions, which may be imposed by e.g. statistical systems of finite size, is naturally defined on a torus. The description of a torus by some modular parameter $\tau \in \mathbb{H}$ (upper half-plane) implies that the torus has to be invariant under the modular group $\Gamma = PSL(2,\mathbb{Z})$. Thus, the partition function is required to be invariant with respect to modular transformations, which upon setting $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$ in the character expressions imposes strong restrictions on the operator content of the theory. Cappelli, Itzykson and Zuber [CIZ87b, CIZ87a] were able to classify all possible partition functions for the minimal models in this manner. Remarkably, these are labeled in terms of the $ADE$ series of simply-laced Lie algebras. A similar $ADE$ classification was done for the non-minimal conformal field theory models with central charge $c = 1$ [Gin88], but for $c > 1$ and non-unitary models with $c < 1$, the results so far are rather sketchy. It is intriguing that many of the character expressions encountered in this thesis will also be shown to feature an interesting connection to these Lie algebras.

Non-unique bases of the Hilbert spaces in two dimensional conformal field theories establish the existence of several alternative character formulae. The Bose-Fermi correspondence introduced above indicates that the characters of two-dimensional quantum field theories can be expressed in a *bosonic* as well as in a *fermionic* way, leading to so-called *bosonic-fermionic q-series identities*. These identities have its roots in the famous *Rogers-Ramanujan identities* [Rog94, Sch17, RR19] (see appendix A)

$$\sum_{n=0}^{\infty} \frac{q^{n(n+a)}}{(q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1-a})(1-q^{5n-4+a})} \tag{0.1}$$

for $a \in \{0,1\}$ with the $q$-deformed Pochhammer symbol defined as

$$(q)_n = \prod_{i=1}^{n}(1-q^n) \quad \text{and by definition} \quad (q)_0 = 1 \quad \text{and} \quad (q)_\infty = \lim_{n\to\infty}(q)_n \ . \tag{0.2}$$

These identities coincide with the two characters of the $\mathcal{M}(5,2)$ minimal model (up to an overall factor $q^\alpha$ for some $\alpha \in \mathbb{Q}$). By using Jacobi's triple product identity (see appendix B or e.g. [And84]), the right hand side of (0.1) can be transformed to give a simple example of a *bosonic-fermionic q-series identity*:

$$\sum_{n=0}^{\infty} \frac{q^{n(n+a)}}{(q)_n} = \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} \left(q^{n(10n+1+2a)} - q^{(5n+2-a)(2n+1)}\right) \ . \tag{0.3}$$

The so-called bosonic expression on the right hand side of (0.3) corresponds to two special cases of the general character formula for minimal models $\mathcal{M}(p,p')$ [RC84]

$$\hat{\chi}^{p,p'}_{r,s} = q^{\frac{c}{24}-h^{p,p'}_{r,s}} \chi^{p,p'}_{r,s} = \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} \left(q^{n(npp'+pr-p's)} - q^{(np+s)(np'+r)}\right) \tag{0.4}$$

with $\hat{\chi}^{p,p'}_{r,s}$ being the normalized character. It has been termed bosonic in [KKMM93b] because it is computed by eliminating null states from the Hilbert space of a free chiral boson [FF83]. The signature of bosonic character expressions is the alternating sign, which reflects the subtraction of null vectors. Furthermore, the factor $(q)_\infty$ keeps track of the free action of the Virasoro 'raising' modes.

In contrast, the fermionic sum representation for a character possesses a remarkable interpretation in terms of an underlying system of quasi-particles. These expressions first occured on the left hand side of the Rogers-Ramanujan identities (0.1). Generalizations have been obtained by George Andrews and Basil Gordon [And74, Gor61] and later on by James Lepowsky and Mirko Primc [LP85]. The most general fermionic expression is regarded to be a linear combination of fundamental fermionic forms. A *fundamental fermionic form* [BMS98, Wel05, DKMM94] is[3]

$$\sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^r \\ \text{restrictions}}} \frac{q^{\vec{m}^t A\vec{m}+\vec{b}^t\vec{m}+c}}{\prod_{i=1}^{j}(q)_i} \prod_{i=j+1}^{r} \begin{bmatrix} g(\vec{m}) \\ m_i \end{bmatrix}_q \tag{0.5}$$

with $A \in M_r(\mathbb{Q})$, $\vec{b} \in \mathbb{Q}^r$, $c \in \mathbb{Q}$, $0 \leq j \leq r$, $g$ a certain linear, algebraic function in the $m_i, 1 \leq i \leq r$, and the $q$-deformed binomial coefficient (the so-called *q-binomial coefficient*) defined as

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{cases} \frac{(q)_n}{(q)_m (q)_{n-m}} & \text{if } \ 0 \leq m \leq n \\ 0 & \text{otherwise} \end{cases} \ . \tag{0.6}$$

It is sometimes also called *universal chiral partition function* for exclusion statistics [BM98, Sch99]. It turns out that (0.5) can be interpreted in terms of a system of $r$ different species

[3] The constant $c$ is not to be confused with the central charge $c_{p,p'}$.

of fermionic quasi-particles with non-trivial momentum restrictions. The bosonic representations are in general unique, whereas there is usually more than one fermionic expression for the same character, giving rise to different quasi-particle interpretations for the same conformal field theory which are conjectured to correspond to different integrable massive extensions of the theory. There are cases for which such correspondences are known. Thus, the different interpretations in terms of quasi-particle systems may be a guide for experimental research. Note that in general, the existence of quasi-particles has been experimentally demonstrated, namely in the case of the fractional quantum Hall effect [SGJE97]. They turned out to be of charge $e/3$, as predicted by Laughlin [Lau83, TSG82].

Furthermore, knowledge of a fermionic character expression and either the theory's effective central charge or a product form of the character results in dilogarithm identities of the form $\frac{1}{\mathrm{L}(1)} \sum_{i=1}^{N} \mathrm{L}(x_i) = d$, where L is the Rogers dilogarithm and $x_i$ and $d$ are rational numbers. It is conjectured [NRT93] that all values of the effective central charges occurring in non-trivial rational conformal field theories can be expressed as one of those rational numbers that consist of a sum of an arbitrary number of dilogarithm functions evaluated at algebraic numbers from the interval $(0, 1)$. Dilogarithm identities in general arise from thermodynamic Bethe ansatz. Conversely, there is also a conjecture [Ter92] that dilogarithm identities corresponding to Bethe ansatz equations $x_i = \prod_{j=1}^{k} (1 - x_j)^{2A_{ij}}$, where $A$ is the inverse Cartan matrix of one of the $ADET$ series of simple Lie algebras (see section 3.5), imply fermionic character expressions of rational conformal field theory characters. Thus, the study of dilogarithm identities arising from conformal field theories gives further insight into the classification of all rational theories.

In addition to the noted minimal models, there exist other theories that have more symmetries than just the Virasoro algebra. They are generated by modes of currents different from the energy-momentum tensor. Possible extensions, which contain the Virasoro algebra as a subalgebra, lead to free fermions, Kac-Moody algebras, Superconformal algebras or $\mathcal{W}$-algebras. In this thesis, the focus is on characters of representations of these extended symmetry algebras (especially of the $\mathcal{W}$-algebras). Specifically, the $\mathcal{W}(2, 2p-1, 2p-1, 2p-1)$ series of so-called triplet algebras [Kau91], which constitute the best understood examples of logarithmic conformal field theory models, is investigated in this thesis. These models have central charges $c_{p,1} = 1 - 6\frac{(p-1)^2}{p}$, $p \geq 2$. For some values of the central charge (when there are fields with integer-spaced dimensions), the existence of fields that lead to logarithmic divergencies in four-point functions is unavoidable [Gur93]. Recently, an attempt at organizing logarithmic theories into families alongside related rational theories has been started [EF06, FGST06b, PRZ06]. Another recent development involving logarithmic conformal field theory is given by Frenkel, Losev and Nekrasov [FLN06] in a work concerning instantons, where it may occur that the Hamiltonian is not diagonalizable. For logarithmic conformal field theories, almost all of the basic notions and tools of (rational) conformal field theories, such as null vectors, (bosonic) character functions, partition functions, fusion rules, modular invariance, have been generalized by now. The main difference to ordinary rational conformal field theories such as the minimal models is the occurence of indecomposable representations.

By contributing a complete set of fermionic sum representations for the characters of the $\mathcal{W}(2, 2p-1, 2p-1, 2p-1)$ logarithmic conformal field theory models with $p \geq 2$ (which are also referred to as the $c_{p,1}$ models), we provide further evidence to answer the question about whether these models, although they lie outside of the usual classification scheme of rational conformal field theories, are nonetheless bona fide theories.

This work is based on our recent article

- *Fermionic Expressions for the Characters of* $c_{p,1}$ *Logarithmic Conformal Field Theories*
  Nucl. Phys. B (2007)
  [www.arxiv.org:hep-th/0611241],

which the author of this thesis wrote in collaboration with fellow diploma student Carsten Grabow under supervision of Michael Flohr and which has been accepted on Jan 26, 2007, for publication in the journal Nuclear Physics B by publisher Elsevier, Amsterdam.[4]

This thesis is organized as follows: In chapter 1, the basics of conformal field theory and $\mathcal{W}$-algebras including the representation theory of the Virasoro algebra are being summarized. Chapter 2 is about modular invariance of systems defined on a torus. The main chapter of this thesis is the third one, where bosonic and fermionic character expressions for various conformal field theories are discussed and the fermionic character expressions for the triplet $\mathcal{W}$-algebra series and the $SU(2)$ WZW model are obtained, which we presented in our recent publication [FGK07]. Chapter 4 is devoted to dilogarithm identities which always arise in a pure conformal field theory context if fermionic character expressions are known. The latter imply an interpretation in terms of fermionic quasi-particles. The quasi-particle content of conformal field theories that admit fermionic expressions for their characters is discussed in chapter 5.

[4] http://www.elsevier.com

# 1. Conformal Symmetry and $\mathcal{W}$-Symmetry

## 1.1. Conformal Symmetry

A short introduction to conformal field theory is provided in this section, which summarizes the basics of what is needed for this thesis. It is based on a conformally invariant Lagrange density. For detailed reviews of conformal field theory basics, the reader is referred to e.g. [Sch95, Sch94, Gab00, Gab03b, Car87, Car89, Nah00, DFMS99]. A lot of interesting conformal field theories do not have a description in terms of a conformally invariant Lagrangian density. In this case, there is another approach, which defines a conformal field theory through its correlation functions. It is called *meromorphic conformal field theory* and is described by *vertex operators*. The theory of vertex operator algebras developed by Richard Borcherds is an attempt at a mathematically rigorous formulation of quantum field theory. For an accessible introduction, the reader is referred to [God89, Kac96, GG00, Gab03a, EFH98]. But the latter approach also has some drawbacks, e.g. it is not suitable in describing low-energy effective field theories. In this thesis, mostly a Langrangian description is used except in the case of the $\mathcal{W}$-algebras.

### 1.1.1. Conformal Transformations

A conformal transformation on some vector space $V$ is a coordinate transformation which acts as a *Weyl transformation*

$$g_{\mu\nu}(x) \to \Omega(x) g_{\mu\nu}(x) \tag{1.1.1}$$

on the metric $g$, where $\Omega(x) \equiv 1$ corresponds to the Poincaré subgroup of the group of conformal transformations. The transformation is called conformal because it preserves angles $\frac{<x,y>}{\sqrt{<x,x><y,y>}}$ $\forall\ x, y \in V$ (if $< .,. >$ is a scalar product on $V$). The conformal croup consists of the following transformations, displayed in global and local form:

| | LOCAL | GLOBAL |
|---|---|---|
| translations | $x^\mu \mapsto x^\mu + \alpha^\mu$ | $x^\mu \mapsto x^\mu + a^\mu$ |
| rotations and/or boosts | $x^\mu \mapsto x^\mu + \lambda^\mu{}_\nu x^\nu$ | $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$ |
| dilations | $x^\mu \mapsto x^\mu + \omega x^\nu$ | $x^\mu \mapsto \Omega x^\mu$ |
| special conformal transf. | $x^\mu \mapsto x^\mu + b^\mu(x^\nu x_\nu) - 2x^\mu(b^\nu x_\nu)$ | $x^\mu \mapsto \frac{x^\mu - b^\mu x^\nu x_\nu}{1-2(b^\mu x_\mu)+(b^\mu b_\mu)(x^\nu x_\nu)}$ |

(1.1.2)

From the infinitesimal transformations, one can calculate the generators of the conformal group and their algebra, called the conformal algebra. The conformal group in a flat space $\mathbb{R}^{p,q}$, where $g_{\mu\nu} = \eta_{\mu\nu}$ with $(\eta_{\mu\nu}) = \mathrm{diag}(\underbrace{-1,-1,\dots,-1}_{q \text{ times}},\underbrace{+1,+1,\dots,+1}_{p \text{ times}})$, is isomorphic to $SO(p+1, q+1)$.

### 1.1.2. Conformal Invariance in Two Dimensions

Given Weyl invariance, we can transform the metric tensor to the form $\Omega(x)\eta_{\mu\nu}(x)$, but in a space with an arbitrary number of dimensions, this would restrict the theory to be non-gravitational. On the contrary, in two dimensions, it is always possible to transform the metric tensor to the form $\Omega(x)\eta_{\mu\nu}(x)$ using a general coordinate transformation. Hence, we can assume the space to be flat and, using a Weyl transformation, rescale it to have metric tensor $g_{\mu\nu} = \eta_{\mu\nu}$. This is called *conformal gauge*.

Suppose we have coordinates $(z^0, z^1)$ on the Euclidean plane. If we perform a general coordinate transformation $\vec{z} \mapsto \vec{w}(\vec{z})$, it will imply for the contravariant metric tensor that it transforms as

$$g^{\mu\nu}(\vec{z}) \mapsto g'^{\mu\nu}(\vec{w}) = \frac{\partial w^\mu}{\partial z^\rho}\frac{\partial w^\nu}{\partial z^\sigma} g^{\rho\sigma} \;, \tag{1.1.3}$$

where $\mu\nu \in \{0,1\}$. In two dimensions, Weyl invariance forces

$$\frac{\partial w^\mu}{\partial z^\rho}\frac{\partial w^\nu}{\partial z^\sigma} g^{\rho\sigma} = \Omega g^{\mu\nu} \tag{1.1.4}$$

and thus leads to the restrictions

$$\frac{\partial w^1}{\partial z^0} = \frac{\partial w^0}{\partial z^1} \quad \text{and} \quad \frac{\partial w^0}{\partial z^0} = -\frac{\partial w^1}{\partial z^1} \;. \tag{1.1.5}$$

These are the Cauchy-Riemann equations. If the range of $z^1$ and $z^2$ is extended to the complex plane[1] using the transformations

$$\begin{aligned} \partial_0 &= \partial_z + \partial_{\bar{z}} & \partial_1 &= \mathrm{i}(\partial_z - \partial_{\bar{z}}) \\ w &= w^0 - \mathrm{i}w^1 & \bar{w} &= w^0 + \mathrm{i}w^1 \\ z &= z^0 - \mathrm{i}z^1 & \bar{z} &= z^0 + \mathrm{i}z^1 \;, \end{aligned} \tag{1.1.6}$$

the Cauchy-Riemann equations become

$$\begin{aligned} &\partial_{\bar{z}} w(z,\bar{z}) = 0 \quad \text{or} \\ &\partial_z \bar{w}(z,\bar{z}) = 0 \;. \end{aligned} \tag{1.1.7}$$

Any holomorphic function $z \mapsto w(z)$ or antiholomorphic function $\bar{z} \mapsto \bar{w}(\bar{z})$ is a solution of these. It is a fundamental theorem of complex analysis that on the complex plane, any holomorphic transformation is conformal. Thus, the conformal group[2] in two dimensions is the set of all analytic[3] maps with the composition of maps as the group multiplication law. Since one can expand an analytic function $z \mapsto w(z)$ into a Laurent series $w(z) = \sum_{n\in\mathbb{Z}} a_n z^n$, an infinite number of parameters is necessary in order to specify a given element of the group. Thus, the local conformal group in two dimensions is infinite-dimensional.

But local transformations are not everywhere well-defined on the Riemann sphere[4] and their inverses may not exist. The subgroup of global conformal transformations, which

[1] The physical subspace is then given by the two-dimensional submanifold given by $z^* = \bar{z}$, called the *real surface*.

[2] It is not really a group because the local transformations mey not be invertible and not be everywhere well-defined. Global transformation, on the other hand, satisfy these requirements.

[3] The terms 'holomorphic' and 'analytic' are used synonymously in the literature.

[4] The *Riemann sphere* is by stereographic 'projection' isomorphic to the complex plane plus the point at infinity, i.e. the *extended complex plane* $\mathbb{C} \cup \{\infty\}$.

are well-defined on the whole Riemann sphere including the point at infinity and which are invertible, is the so-called *special conformal group*. The elements of this group are the *Möbius transformations*

$$f(z) = \frac{az+b}{cz+d} \quad a,b,c,d \in \mathbb{C} \quad ad-bc=1 \ . \tag{1.1.8}$$

Since we can associate a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.1.9}$$

with it, where the composition of maps corresponds to matrix multiplication, the global conformal group is isomorphic to $SL(2,\mathbb{C})$, the group of invertible $2\times 2$ matrices with unit determinant whose entries are complex numbers and which in turn is isomorphic to $SO(3,1)$, in accordance with section 1.1.1.

### 1.1.3. The Generators of Conformal Transformations in Two Dimensions

From the effect of an infinitesimal holomorphic coordinate transformation $z \mapsto z+\epsilon(z)$ on a field $\Phi(z)$, one can read off, when expanding the function $\epsilon$ in Laurent modes, that the generators for the infinite-dimensional conformal group are given by

$$\ell_n = -z^{n+1}\partial_z \tag{1.1.10}$$

and analog for antiholomorphic transformation. These generators obey the so-called *Witt algebra*

$$\begin{aligned} [\ell_n,\ell_m] &= (n-m)\ell_{n+m} \\ [\bar{\ell}_n,\bar{\ell_m}] &= (n-m)\bar{\ell}_{n+m} \\ [\ell_n,\bar{\ell}_m] &= 0 \ . \end{aligned} \tag{1.1.11}$$

Obviously, $\ell_{-1}$, $\ell_0$ and $\ell_1$ form a subalgebra, corresponding to the global conformal group: $\ell_{-1} = -\partial_z$ generates translations on the complex plane, $\ell_0 = -z\partial_z$ scale transformations and rotations and $\ell_1 = -z^2\partial_z$ special conformal transformations. $\ell_0+\bar{\ell}_0$ generates dilations on the real surface and $\mathrm{i}(\ell_0-\bar{\ell}_0)$ rotations.

### 1.1.4. Primary Fields

In general, fields $\Phi(z,\bar{z})$ transform non-trivial under a conformal tranformation $z \mapsto w(z)$. But in the simplest case, this transformation behavior is given by just a scalar prefactor:

$$\Phi(z,\bar{z}) \mapsto \left(\frac{\partial w}{\partial z}\right)^h \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\bar{h}} \Phi\left(w(z),\bar{w}(\bar{z})\right) \ . \tag{1.1.12}$$

If this transformation behavior holds for the field $\Phi(z,\bar{z})$, then $\Phi$ is called *primary*, and $h$ and $\bar{h}$ are called *conformal weights* of the field $\Phi$. If this transformation behavior holds only for the $SL(2,\mathbb{C})$ subgroup, then the field is called *quasi-primary*. If a field is not (quasi-)primary, it is usually called *secondary*. Derivatives of primary fields are secondary. But secondary fields can also be primary, as we will see later. For infinitesimal conformal transformations $w(z) = z + \epsilon(z)$, (1.1.12) is of the form

$$\delta_\epsilon \Phi(w,\bar{w}) = h\partial_w\epsilon(w)\Phi(w,\bar{w}) + \epsilon(w)\partial_w\Phi(w,\bar{w}) + \bar{h}\partial_{\bar{w}}\bar{\epsilon}(\bar{w})\Phi(w,\bar{w}) + \bar{\epsilon}(\bar{w})\partial_{\bar{w}}\Phi(w,\bar{w}) \ . \quad (1.1.13)$$

On the infinite plane, the holomorphic and antiholomorphic sectors of the conformal field theory completely decouple. This situation will change when the theory is defined on arbitrary surfaces. But since the sectors do not interfere on the infinite plane, they may be studied as independent theories on their own in that case, because the correlation functions factorize into holomorphic and antiholomorphic factors. While working on the infinite plane, the antiholomorphic dependence is thus where possible suppressed in the following. This will shorten the notation and a generalization will be obvious.

### 1.1.5. Radial Quantization and Operator Product Expansion

To quantize the theory, such that the conformal fields become operator valued distributions in some Hilbert space $\mathcal{H}$, we seek a representation of $\ell_n \in \mathrm{Diff}(S^1)$ by some operators $L_n$ acting on $\mathcal{H}$ such that

$$\delta_n \Phi_h(z) = [L_n, \Phi_h(z)] \ . \quad (1.1.14)$$

When the space dimension is being compactified, i.e. periodic boundary conditions are imposed, which results in the momenta being quantized, infrared divergencies can be avoided. In that case, one is no longer working on the plane but on a cylinder. A cylinder can be mapped back conformally to the complex plane, so that use can still be made of complex analysis. The map which does the trick is

$$w \mapsto z = \mathrm{e}^w \quad (1.1.15)$$

with $z = z^0 + \mathrm{i}z^1$ and $w = w^0 + \mathrm{i}w^1$, where $w^0$ is the time coordinate and $w^1$ is the space coordinate on the cylinder. This map is obviously conformal. It has the effect of an equal-time slice through the cylinder being turned into a circle around the origin of the complex plane, just like an annual ring of a tree, as displayed in figure 1.1. On the complex plane, the space coordinate is consequently given by an angle, whereas the time coordinate is given by the distance from the origin: Infinite past and infinite future correspond to $z = 0$ and $z = \infty$, respectively.

In general, according to *Noether's theorem*, every symmetry corresponds to a conserved current $j^\mu$ with $\partial_\mu j^\mu = 0$. The conserved charge is then given by

$$Q = \int d^{d-1}x j^0(\vec{x}, t) \ , \quad (1.1.16)$$

the integral over the $d-1$ space dimensions, and is the generator of infinitesimal symmetry variations of the field $\Phi$,

$$\delta_\epsilon = \epsilon[Q, \Phi] \ . \quad (1.1.17)$$

In two dimensions, one thus has to integrate over a circle of constant radius. However, in complex analysis, the integral of $w(z)$ around a simple closed curve, where $w(z)$ is a single-valued function which is analytic inside and on that simple closed curve except at a number of singularities is only determined by the enclosed singularities and does, aside from that, not

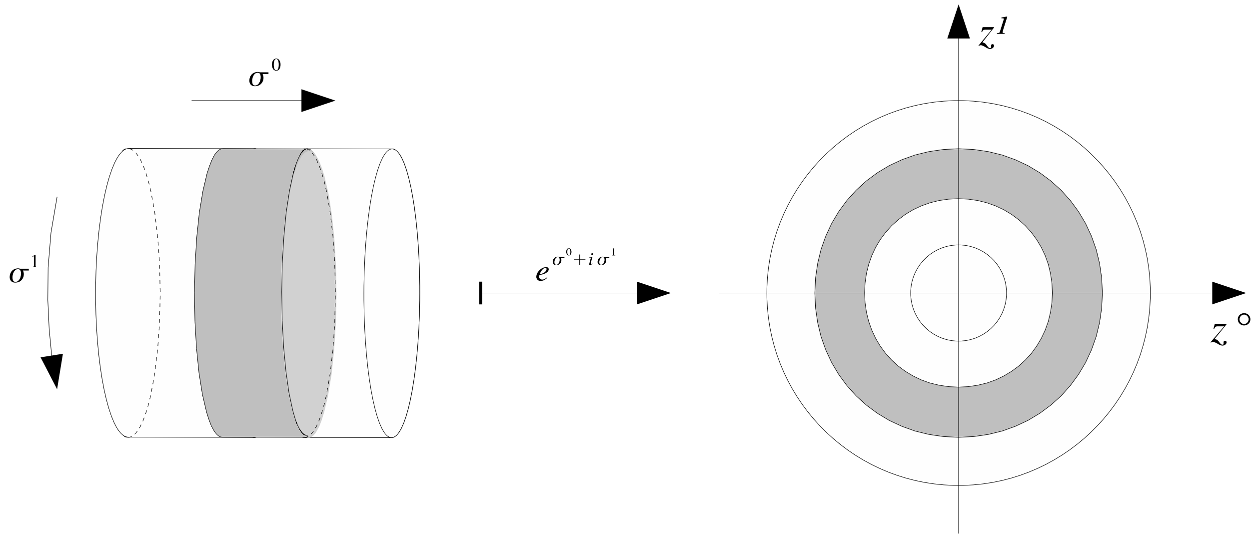



**Figure 1.1.:** Conformal mapping from the cylinder to the complex plane

depend on the chosen contour of integration. Because of Weyl invariance, the stress-energy tensor is traceless, which means in complex coordinates $T_{z\bar{z}} = 0$, and because of general coordinate invariance, it is conserved such that $\partial_{\bar{z}} T_{zz} = \partial_z T_{\bar{z}\bar{z}} = 0$. Accordingly, $T_{zz} =: T$ is holomorphic and $T_{\bar{z}\bar{z}} =: \bar{T}$ is antiholomorphic.[5] The generator of holomorphic conformal transformations on the complex plane is thus given by

$$Q = \frac{1}{2\pi \mathrm{i}} \oint dz T(z)\epsilon(z) \ . \tag{1.1.18}$$

This integral is only a formal expression and needs to be evaluated with another field in the contour of integration. Inserting (1.1.18) into (1.1.17) yields

$$\delta_\epsilon \Phi(z) = \frac{1}{2\pi \mathrm{i}} \oint [dz T(z)\epsilon(z), \Phi(\zeta)] \ , \tag{1.1.19}$$

where $[.,.]$ as usual denotes the equal-time commutator. When the cylinder is being mapped to the complex plane, the time ordered product becomes the *radially ordered product*

$$R(\Phi(z)\Psi(w)) = \begin{cases} \Phi(z)\Psi(w) & \text{if} \quad |z| > |w| \\ \Psi(w)\Phi(z) & \text{if} \quad |w| > |z| \end{cases} . \tag{1.1.20}$$

Cauchy's theorem suggests that the commutator may be defined as the difference between contour integrals of the radially ordered product around 0

$$\delta_\epsilon \Phi(z) = \frac{1}{2\pi \mathrm{i}} \left( \oint_{|z|>|w|} - \oint_{|z|<|w|} \right) (dz \epsilon(z) R(T(z)\Phi(w))) \tag{1.1.21}$$

$$= \frac{1}{2\pi \mathrm{i}} \oint_w (dz \epsilon(z) R(T(z)\Phi(w))) \ , \tag{1.1.22}$$

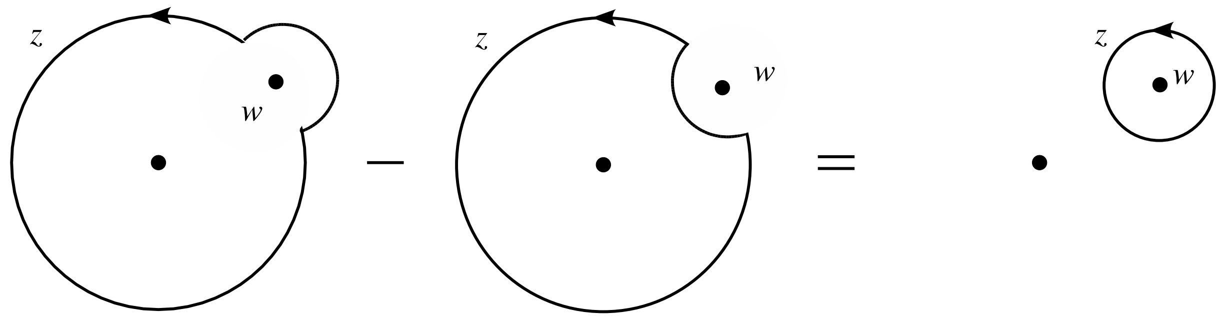



**Figure 1.2.:** The resulting contour of integration for the radially ordered product

which include and exclude the point $w$, repectively. The resulting contour is displayed graphically in figure 1.2. This ought to equal (1.1.13), which is the case if

$$R(T(z)\Phi(w,\bar{w})) = \frac{h}{(z-w)^2}\Phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\Phi(w,\bar{w}) + \dots \ . \tag{1.1.23}$$

Thus, this *operator product expansion* defines $\Phi$ to be primary.

The product of field operators evaluated close together diverges. The general short-distance operator product of two fields, which encodes the singularities that occur when operators approach one another, is given by

$$R(A(z)B(w)) = \sum_{n=-\infty}^{N} \frac{(AB)_n(w)}{(z-w)^n} \ . \tag{1.1.24}$$

In following operator products, the radial ordering symbol will be omitted, since an operator product expansion is always intended to correspond to a radially ordered product, and the non-divergent part will be omitted, too.

### 1.1.6. The Central Charge and the Virasoro Algebra

Similarly, the operator product of the stress-energy tensor with itself may be determined as

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) \ . \tag{1.1.25}$$

The constant $c$ is called *central charge*. Thus, $T(z)$ is not primary. If the first term in the expansion, which is called *conformal anomaly*, did not occur, then $T(z)$ would be a primary field of weight 2. Physically, the central charge turns out to be proportional to the *Casimir energy*, the change in the vacuum energy density that arises from the periodic boundary conditions on the cylinder. Indeed,

$$T_{\text{cyl}}(w) = z^2 T_{\text{plane}}(z) - \frac{c}{24} \ , \tag{1.1.26}$$

[5]'Holomorphic' is commonly used different than in the mathematics literature. It does not mean the absence of singularities, but only the dependence on $z$ and not on $\bar{z}$, and vice versa for the term 'antiholomorphic'.

so that

$$\langle T_{\rm cyl}(w)\rangle = -\frac{c}{24} \tag{1.1.27}$$

if the vacuum energy density $\langle T_{\rm plane}\rangle$ is assumed to be zero on the plane. The central charge is determined solely by the short-distance behavior and is a measure for the extensive degrees of freedom of the systems. For example, $c = 1$ corresponds to the case of the free boson, $c = \frac{1}{2}$ to the free fermion, $c = -26$ to the reparametrization ghost and $c = -2$ to the simple ghost system. It adds up if decoupled systems are put together. A Laurent expansion of the stress-energy tensor may be defined as

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n\ , \quad \bar{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{z}^{-n-2}\bar{L}_n\ , \tag{1.1.28}$$

with the inversion

$$L_n = \oint \frac{dz}{2\pi \mathrm{i}} z^{n+1} T(z)\ , \quad \bar{L}_n = \oint \frac{d\bar{z}}{2\pi\mathrm{i}} \bar{z}^{n+1}\bar{T}(\bar{z})\ . \tag{1.1.29}$$

The commutator of these modes can then again be computed using contour integrals, which leads to the *Virasoro algebra* $\mathcal{V}$

$$[L_n, L_m] = (n-m)L_{m+n} + \frac{c}{12} n(n+1)(n-1)\delta_{n+m,0} \quad \forall n,m\in\mathbb{Z} \tag{1.1.30}$$

$$[\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{m+n} + \frac{\bar{c}}{12} n(n+1)(n-1)\delta_{n+m,0} \quad \forall n,m\in\mathbb{Z} \tag{1.1.31}$$

$$[L_n, \bar{L}_m] = 0\ . \tag{1.1.32}$$

In general, an operator product expansion algebra of currents is equivalent to the commutator algebra of their Laurent modes. Note that for $c = \bar{c} = 0$, this reduces to the classical Witt algebra. Note also that $L_{-1}$, $L_0$ and $L_1$ form a subalgebra and generate the global conformal group $SL(2,\mathbb{C})$ which thus remains an exact symmetry group despite the central charge term. It is the most general central extension which constains $sl(2,\mathbb{C})$ (Moebius) as subalgebra. $L_0 + \bar{L}_0$ generates dilations, which are time translations in radial quantization, hence $L_0 + \bar{L}_0$ is proportional to the *Hamiltonian* of the system.

The commutator of the modes of the stress-energy tensor with primary fields is given by

$$\delta_n \Phi_h(z) = [L_n, \Phi_h(z)] = h(n+1)z^n\Phi_h(z) + z^{n+1}\partial_z\Phi_h(z)\ . \tag{1.1.33}$$

### 1.1.7. Correlation Functions

Vacuum expectation values of primary fields are restricted by conformal symmetry. Using the abbreviation $z_{ij} = z_i - z_j$, the $SL(2,\mathbb{C})$ subgroup forces a two-point function to be of the form

$$G^{(2)}(z_i,\bar{z}_i) = \langle \Phi_1(z_1,\bar{z}_1)\Phi_2(z_2,\bar{z}_2)\rangle = C_{12}(z_{12})^{-2h}(\bar{z}_{12})^{-2\bar{h}} \quad \text{if} \quad \begin{cases} h_1 = h_2 = h \\ \bar{h}_1 = \bar{h}_2 = \bar{h} \end{cases}, \tag{1.1.34}$$

where $C_{12}$ could be absorbed into a normalization of the primary fields, and a three-point function to be of the form

$$G^{(3)}(z_i,\bar{z}_i) = \frac{C_{123}}{z_{12}^{h_{123}} z_{23}^{h_{231}} z_{13}^{h_{312}} \bar{z}_{12}^{\bar{h}_{123}} \bar{z}_{23}^{\bar{h}_{231}} \bar{z}_{13}^{\bar{h}_{312}}} \text{with} \begin{cases} h_{ijk} = h_i + h_j - h_k \\ \bar{h}_{ijk} = \bar{h}_i + \bar{h}_j - \bar{h}_k \end{cases}. \tag{1.1.35}$$

This correlation function does in principle only depend on a single constant, as one can always take three reference points $z_1 = \infty$, $z_2 = 1$ and $z_3 = 0$ for the evaluation and then determine the conformal mapping of these three points to arbitrary $z_i$ by solving (1.1.8) simultaneously for all $z_i$. The constants $C_{123}$ can not be fixed by $SL(2,\mathbb{C})$ invariance and are called *structure constants* of the conformal field theory. But due to the fact that a global conformal transformation only allows us to fix three coordinates, the four-point function still depends on one coordinate, the so-called *cross ratio*

$$x = \frac{z_{12}z_{34}}{z_{13}z_{24}}\ , \quad \bar{x} = \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{13}\bar{z}_{24}}\ , \tag{1.1.36}$$

and is given by

$$G^{(4)}(z_i,\bar{z}_i) = f(x,\bar{x}) \prod_{\substack{i<j \\ i,j\in\{1,2,3,4\}}} z_{ij}^{\frac{h}{3}-h_i-h_j}\, \bar{z}_{ij}^{\frac{\bar{h}}{3}-\bar{h}_i-\bar{h}_j} \quad \text{with} \begin{cases} h = \sum_{i=1}^{4} h_i \\ \bar{h} = \sum_{i=1}^{4} \bar{h}_i \end{cases} . \tag{1.1.37}$$

In general, an $n$-point function depends on $n-3$ cross ratios.

The *conformal Ward identities*

$$\delta_{\epsilon,\bar{\epsilon}} G^{(N)}(z_i,\bar{z}_i) = -\frac{1}{2\pi \mathrm{i}} \oint_{\{(z_i,\bar{z}_i)\}} dw\,\epsilon(w) \langle T(w)\left(\Phi_{h_1,\bar{h}_1}(z_1,\bar{z}_1)\cdots\Phi_{h_N,\bar{h}_N}(z_N,\bar{z}_N)\right)\rangle\ , \tag{1.1.38}$$

which employ the extra information obtained by inserting the generator of infinitesimal conformal transformations if the full conformal group in two dimensions is used instead of $SL(2,\mathbb{C})$ only, then put further constraints on correlation functions and allow to express correlators of descendants of primary fields in terms of correlators of their primary fields. Thus, an infinite set of fields is grouped into so-called *conformal families* of their primary field. Several other constraints like channel-duality and null-state decoupling then in principle enable one to compute all correlation functions of certain two-dimensional theories, which are hence called exactly solvable. For the subgroup of global conformal transformations $SL(2,\mathbb{C})$, (1.1.38) implies (1.1.34) and (1.1.35).

## 1.2. Representation Theory of the Virasoro Algebra

An introduction to the representation theory of the Virasoro algebra is provided. Some definitions are in order. The representation theory of the Virasoro algebra $\mathcal{V}$ is mainly analogous to the representation theory of finite-dimensional Lie algebras (for an introduction to the latter see e.g. [Hum72, Geo99]).

### 1.2.1. The Space of States

The vacuum expectation value of the stress-energy tensor $\langle T(z)\rangle$ is demanded to be well-defined, i.e. regular, as $z \to 0$. If the vacuum, which is assumed to be $SL(2,\mathbb{C})$ invariant, is regarded as some state of an Hilbert space $\mathcal{H}$ and $T(z)$ as some linear operator acting on that Hilbert space, then also the Laurent modes of $T(z)$ become linear operators. The regularity condition then implies that

$$L_m|0\rangle = 0 \quad \text{for all } m \geq -1 \ . \tag{1.2.1}$$

One may then consider the action of a primary field operator on this vacuum state. From (1.1.33), it follows that $L_0\Phi_h(z=0)|0\rangle = h\Phi_h(0)|0\rangle$, i.e. $\Phi_h(0)|0\rangle$ is an eigenstate of $L_0$ to the eigenvalue $h$. In general, there is an isomorphism $V$ between the fields in the theory and the states in the Hilbert space $\mathcal{H}$ induced by the mapping

$$V(\Phi_h) = \lim_{z\to 0}\Phi_h(z)|0\rangle \ , \tag{1.2.2}$$

and we define $\lim_{z\to 0}\Phi_h(z)|0\rangle \equiv |h\rangle$ or, in general, $\lim_{z\to 0}\Phi_{h,\bar{h}}(z,\bar{z})|0\rangle \equiv |h,\bar{h}\rangle$ as the *asymptotic in-state* corresponding to the *asymptotic field* $\Phi_h(z=0)$, since the interaction is assumed to be attenuated in the infinite past. In wanting to construct a Hilbert space of states, a scalar product has to be defined. Therefore, the *asymptotic out-state*

$$\langle h| := |h\rangle^\dagger \ , \tag{1.2.3}$$

corresponding to the infinite future, is defined as the Hermitean conjugate of the asymptotic in-state on the *real surface* $\bar{z} = z^*$. This Hermitean conjugation has to be defined as[6]

$$\begin{aligned} |h,\bar{h}\rangle^\dagger &= \lim_{z,\bar{z}\to 0}\langle 0|\left(\Phi_{h,\bar{h}}(z,\bar{z})\right)^\dagger \\ &:= \lim_{z,\bar{z}\to 0}\bar{z}^{-2h}z^{-2\bar{h}}\langle 0|\Phi_{h,\bar{h}}(1/\bar{z},1/z) = \lim_{w,\bar{w}\to\infty}\bar{w}^{2h}w^{2\bar{h}}\langle 0|\Phi_{h,\bar{h}}(\bar{w},w) \end{aligned} \tag{1.2.4}$$

so that when defining the mode expansion of $\Phi_{h,\bar{h}}(z,\bar{z})$ as

$$\Phi_{h,\bar{h}}(z,\bar{z}) = \sum_{m,n\in\mathbb{Z}}\Phi_{m,n}z^{-m-h}\bar{z}^{-n-\bar{h}} \ , \tag{1.2.5}$$

then

$$\begin{aligned} \Phi_{h,\bar{h}}(z,\bar{z})^\dagger &= \bar{z}^{-2h}z^{-2\bar{h}}\Phi_{h,\bar{h}}(1/\bar{z},1/z) \\ &= \bar{z}^{-2h}z^{-2\bar{h}}\sum_{m,n\in\mathbb{Z}}\Phi_{m,n}\bar{z}^{m+h}z^{n+\bar{h}} \\ &= \sum_{m,n\in\mathbb{Z}}\Phi_{-m,-n}\bar{z}^{-m-h}z^{-n-\bar{h}} \ . \end{aligned} \tag{1.2.6}$$

It follows that this definition of Hermitean conjugation on the real surface implies

$$\Phi_{-m,-n} = (\Phi_{m,n})^\dagger \ . \tag{1.2.7}$$

If on the real surface, i.e. (1.2.7) holds, and the representation is a Hilbert space, i.e. if it has a scalar product (implying that the norm will be positive definite), then the theory will be unitary.

[6]Euclidean time $\tau = it$ has to be reversed in order to keep the Minkowskian time $t$ unchanged during Hermitean conjugation.

$$\begin{tikzcd}
 & \mathcal{U} \arrow[dr, "\Phi"] & \\
\mathcal{V} \arrow[ur, "i"] \arrow[rr, "j"] & & \mathcal{A}
\end{tikzcd}$$

**Figure 1.3.:** The idea of the universal enveloping algebra displayed schematically

### 1.2.2. The Universal Enveloping Algebra

The term *module* is used synonymously with the term representation. For a general Lie algebra $\mathcal{L}$, a vector space $V$, endowed with an operation $\mathcal{L} \times V \to V$, $(x, v) \mapsto xv$, is called an *$\mathcal{L}$-module* if the following conditions are satisfied:

$$(ax + by)v = a(xv) + b(yv) \tag{1.2.8}$$
$$x(av + bw) = a(xv) + b(xw) \tag{1.2.9}$$
$$[x, y]v = x(yv) - y(xv) \quad (x, y \in \mathcal{L}, v, w \in V, a, b \in \mathbb{C}). \tag{1.2.10}$$

$\mathcal{L}$ itself is an $\mathcal{L}$-module for the adjoint representation.

The representation space decomposes into irreducible representations of the Virasoro algebra which are labeled by the $L_0$ eigenvalue of highest weight states if $L_0$ acts diagonally. For certain symmetry algebras, e.g. certain extensions of the Virasoro algebra called $\mathcal{W}$-algebras, $L_0$ may not be diagonalizable. In that case, it is not possible to decompose the representation space into irreducible representations of the extended symmetry algebra. There exist indecomposable representations. However, the simple case will be studied first. An $\mathcal{L}$-module $V$ is called *irreducible* if it contains no proper submodule (i.e. other than itself and 0). It is called *indecomposable* if there exist no submodules $V_1, V_2 \subset V$ such that $V = V_1 \oplus V_2$. $V$ is called *completely reducible* if $V$ is a direct sum of irreducible $\mathcal{L}$-submodules. The study of representations becomes easier when extending to the unital associative algebra:

The *universal enveloping algebra* of $\mathcal{V}$ is a pair $(\mathcal{U}, i)$ with

(i) $\mathcal{U}$ is a unital (i.e. contains $\mathbb{1}$) associative $\mathbb{C}$-algebra.

(ii) The embedding $i : \mathcal{V} \to \mathcal{U}$ is linear with $i([x, y]) = i(x)i(y) - i(y)i(x) \;\; \forall x, y \in \mathcal{V}$.

(iii) For every unital associative $\mathbb{C}$-algebra $\mathcal{A}$ and every linear map $j : \mathcal{V} \to \mathcal{A}$ with $j([x, y]) = j(x)j(y) - j(y)j(x) \;\; \forall x, y \in \mathcal{V}$, there exists exactly one algebra homomorphism $\Phi : \mathcal{U} \to \mathcal{A}$ with $\mathbb{1}_{\mathcal{U}} \mapsto \mathbb{1}_{\mathcal{A}}$ such that $\Phi \circ i = j$.

Thus, every $\mathcal{V}$-module is mapped to a unique $\mathcal{U}$-module as displayed diagramatically in figure 1.2.2. The universal enveloping algebra exists. It can be directly constructed from $\mathcal{V}$ as the quotient of the tensor algebra $T(\mathcal{V}) = \bigoplus_{n=0}^{\infty} \mathcal{V}^{\otimes n}$, which is associative and has a unit element, of the vector space underlying $\mathcal{V}$ and the two-sided ideal in $T(\mathcal{V})$ which is given by $I = \langle \{[x, y] - (x \otimes y - y \otimes x) \mid x, y \in \mathcal{V}\} \rangle$, $\langle \cdot \rangle$ denoting the span and $[\cdot, \cdot]$ the algebra operation of $\mathcal{V}$. Furthermore, the universal enveloping algebra is unique up to isomorphism.

A *basis* of the universal enveloping algebra of $\mathcal{V}$ is given by the *Poincaré-Birkhoff-Witt basis*. It consists of all ordered monomials (i.e. words) of Virasoro generators and is given by

$$\left\{ L_{-n_1} L_{-n_2} \cdots L_{-n_k} C^i \mid n_j \in \mathbb{Z},\; k, i \in \mathbb{Z}_{\geq 0},\; n_i \geq n_{i+1} \right\} \; . \tag{1.2.11}$$

A basis of the level $N$ of $\mathcal{U}$ is given by the subset of the PBW basis for which the condition $\sum_{j=1}^{k} n_j = N$ holds. Its span will be called $\mathcal{U}_N$. Since the central element $C$ is proportional to the identity according to Schur's Lemma because it commutes with every other operator in the algebra, these bases are further reduced, because monomials which differ only by the number of $C$ operators are being identified.

### 1.2.3. Highest Weight Modules

From (1.2.1) and (1.1.33), one deduces that

$$L_n|h\rangle = 0 \quad \text{for all } n > 0 \ . \tag{1.2.12}$$

This is the condition for a highest weight state of a Virasoro representation. Thus, highest weight states correspond to primary fields. In the following, $\mathcal{U}^{\pm}$ denotes the universal enveloping algebra of the $\mathcal{V}$-subalgebra $\mathcal{V}^{\pm}$ that is given by the span over $\{L_n \mid n \gtrless 0\}$.

If $V$ denotes a $\mathcal{V}$-module, then a state $|\chi\rangle \in V$ is called *singular vector* if $\mathcal{U}^{+}v = 0$. Furthermore, $V$ is called *highest weight module* if it contains a singular vector $|h\rangle$ with

(i) $L_0|h\rangle = h|h\rangle, \quad h \in \mathbb{C}$

(ii) $C|h\rangle = c|h\rangle, \quad c \in \mathbb{C}$

(iii) $V = \mathcal{U}^{-}|h\rangle$

The state $|h\rangle$ is then called *highest weight state* and $h$ is called *highest weight* of the representation. The states obtained from $|h\rangle$ by application of $L_{-n_1}L_{-n_2}\cdots L_{-n_k}$, $n_i \geq n_{i+1}$, $k, n_i \in \mathbb{Z}_{>0} \forall i \in \{1,\ldots,k\}$ are called *descendant states*. It may occur that one of the descendants is also a singular vector. In that case, that descendant gives rise to a full Virasoro subrepresentation of $V$, i.e. a subset of $V$ that is closed under the action of the Virasoro generators and consists of the descendant singular vector and its descendants. In this case, the representation is not a Hilbert space, since a scalar product, which would be demanded to be positive definite, does not exist in that case.

### 1.2.4. Verma Modules

There is a mathematically defined abstract object called Verma module which can be mapped onto any highest weight module: A highest weight module $V$ with highest weight $h$ and highest weight state $|h_V\rangle$ is called *Verma module* if it has the universal property that for any highest weight module $W$ with highest weight $h$, highest weight vector $|h_W\rangle$ and the same central charge $c$, there is a unique $\mathcal{V}$-homomorphism $V \to W$ mapping $|h_V\rangle$ to $|h_W\rangle$.

The Verma module $V(h,c)$ for any given $c, h \in \mathbb{C}$ exists and is unique (of course, up to $\mathcal{V}$-isomorphism) with a basis given by

$$\{L_{-n_1}\cdots L_{-n_k}|h\rangle \mid k \in \mathbb{Z}_{\geq 0},\ n_{i+1} \geq n_i\} \ , \tag{1.2.13}$$

where $|h\rangle$ is the highest weight vector of $V(h,c)$. The existence may be proven by construction, and the uniqueness then follows from the universal property.

A Verma module is called *degenerate* if it contains a proper submodule, i.e. other than 0 and itself. The *maximal proper submodule* is given by the sum of all proper submodules. If one builds the quotient module of the Verma module $V(h,c)$ by its maximal proper submodule, one obtains an *irreducible highest weight module* $M(h,c)$.

### 1.2.5. The Submodule Structure of Verma Modules

One can analyze the submodule structure using the so-called *Shapovalov form* $\langle .,. \rangle$, a symmetric bilinear form on $V(h,c)$. This bilinear form does not have to be a scalar product, since it is not required to be positive definite. The highest weight vector $|h\rangle$ of $V(h,c)$ satifies $\langle |h\rangle, |h\rangle\rangle = 1$ and $\langle |h\rangle, |\xi\rangle\rangle = 0\ \forall |\xi\rangle \in V_{k>0}$. The *dual module* is built on the *dual highest weight vector* $(|h\rangle)^\dagger \in V(h,c)^*$ by action of

$$\left\{ \left(L_{-k_1}L_{-k_2}\cdots L_{-k_j}\right)^\dagger := L_{k_j}L_{k_{n-1}\cdots L_{k_1}} \mid j \in \mathbb{Z}_{\geq 0},\ k_{i+1} \leq k_i \right\} .$$

If the Shapovalov form is restricted to a certain level $l \geq 0$, one can study the determinant of the matrix which is obtained by inserting all possible combinations of basis vectors of $V(h,c)_l$ into $\langle .,. \rangle_l$. Because it may occur that a Verma module is degenerate, i.e. reducible, it possibly contains singular vectors. The Shapovalov form of a singular vector $|\chi\rangle$ with itself is zero. Moreover, $|\chi\rangle$ is orthogonal to the whole Verma module because

$$\langle \chi | L_{-k_1}L_{-k_2}\cdots L_{-k_j}|h\rangle = \langle h|L_{k_j}L_{k_{j-1}\cdots L_{k_1}}|\chi\rangle^* = 0 \ . \tag{1.2.14}$$

Obviously, this requires a sesquilinear form $\langle .|. \rangle$. A sesquilinear form is easily constructed from the Shapovolov form. The matrix $\langle .,. \rangle_l$ is unaffected by that change and equals $\langle .|. \rangle_l$. If one has a sesquilinear form and imposes the condition $L_n^\dagger = L_{-n}$, the representation will be unitary if it is a Hilbert space, i.e. if the sesquilinear Shapovalov form is positive definite and thus a scalar product, which is the case for an irreducible representation $M(c,h)$. The stress-energy tensor $T$ will then furthermore be Hermitean. Studying the sesquilinear form of $L_{-n}|h\rangle$ with itself,

$$\langle h|L_nL_{-n}|h\rangle = \langle h|(L_{-n}L_n + 2nL_0 + \frac{c}{12}n(n+1)(n-1))|h\rangle \tag{1.2.15}$$
$$= (2nh + \frac{c}{12}n(n+1)(n-1))\langle h|h\rangle \ , \tag{1.2.16}$$

it turns out that it can only be non-negative for all $n \geq 0$ if $c \geq 0$ and $h \geq 0$. Therefore, a representation of highest weight $h$ for which either $c < 0$ or $h < 0$ is always non-unitary. Moreover, one can show that representations with $c \geq 1$ and $h \geq 0$ are always unitary [Lan88, FSQ86]. On $M(h,c)$, the sesquilinear Shapovalov form is non-degenerate and hence a scalar product. This implies that one can read off from the determinant of $\langle .|. \rangle_l$ for which levels the Verma module $V(h,c)$ has a non-trivial intersection with the maximal proper submodule. There is an explicit formula for this determinant, which was stated by Victor G. Kač [Kac79] and proven bei Boris L. Feigin and Dimitrij Borisovich Fuks [FF82] in terms of a free-field construction (see e.g. [Fel89]). It is aptly called *Kač determinant*. The Kač determinant of the matrix of $\langle .|. \rangle_l$ on $V(h,c)_l$ is given by

$$\det{}_l(h,c) = K_l \prod_{\substack{r,s\in\mathbb{N} \\ rs\leq l}} (h - h_{r,s}(c))^{p(l-rs)} \quad \text{with} \tag{1.2.17}$$

$$h_{r,s}(c) = \frac{1}{48}((13-c)(r^2+s^2) \pm \sqrt{(c-1)(c-25)}(r^2-s^2) - 24rs - 2 + 2c) \ , \tag{1.2.18}$$

where $p(n)$ denotes the number of additive *partitions* (see also appendix A) of an integer $n$ into arbitrary many integers greater than or qual to one with generating function

$$\prod_{n\in\mathbb{Z}_{\geq 1}} (1-q^n)^{-1} = \sum_{n\in\mathbb{Z}_{\geq 0}} p(n)q^n \ , \tag{1.2.19}$$

and where $K_l$ are nonvanishing constants (depending on the choice of base). A change of the branch taken in (1.2.18) just amounts to an interchange of $r$ and $s$. Hence, the first singular vector, or null state, in a reducible Verma module with a highest weight $h$ that can be parametrized by $r$ and $s$ as in (1.2.18) occurs at level $l = rs$. The number of additive partitions into integer parts greater than zero of an integer smaller than one is defined to be zero.

Introducing the parameterization

$$c = 1 - \frac{6}{m(m+1)} \quad (m \in \mathbb{C}) \tag{1.2.20}$$

for the central charge, $h_{r,s}(c)$ may be rewritten as

$$h_{r,s}(m) = \frac{\left((m+1)r - ms\right)^2 - 1}{4m(m+1)} \ . \tag{1.2.21}$$

It can be shown that theories with $0 \leq c < 1$ are only unitary if $m \in \mathbb{Z}_{\geq 2}$ and if $1 \leq r \leq m-1$, $1 \leq s < r$ [FQS84, FSQ86, Lan88]. Figure 1.4 displays a plot of the unitary representations in a section of the $(h, c)$ plane, created with the mathematics software package Maple. It is illuminating to also plot different sections.

With the Kač determinant, Feigin and Fuks determined the Verma modules that can be embedded in a given $V(h, c)$ [FF83]. The maximal proper submodule of $V(h, c)$ is then the sum of these embedded modules.

Reparameterizing the central charge by

$$c = 1 - 24k \tag{1.2.22}$$

and inserting this into (1.2.18) leads to

$$h_{r,s} = -k + \frac{1}{4}((2k+1)(r^2+s^2) + 2\sqrt{k(k+1)}(r^2-s^2) - 2rs) \ . \tag{1.2.23}$$

If there exist no $r, s \in \mathbb{Z}_{\geq 1}$ such that $h = h_{r,s}$, then $V(h, c)$ itself is irreducible. In case $V(h, c)$ is not irreducible, it is degenerate. The degenerate representations $\mathcal{V}$ can be classified by $k$ and $k' := \sqrt{k(k+1)}$, where $V_{r,s} := V(h_{r,s}, c)$:

**Case 1 ($k, k' \in \mathbb{Q}$):** $k = \frac{(p-p')^2}{4pp'}$ with $p, p' \in \mathbb{Z}_{\geq 1}$ coprime and therefore $c_{p,p'} = 1 - 6\frac{(p-p')^2}{pp'}$. Moreover, $h_{r,s} \in \mathbb{Q} \forall r, s \in \mathbb{Z}$. There are three different subcases:

**Minimal Models ($p > p' > 1$):** There is an infinite number of singular vectors and only finitely many highest weight representations of the Virasoro algebra with $h_{r,s} = \frac{(pr-p's)^2-(p-p')^2}{4pp'}$, restricted to $1 \leq r \leq p'-1$, $1 \leq s \leq p-1$ because of $h_{r,s} = h_{r+p',s+p}$. One has the following embedding structure:

$$\begin{array}{ccccccccc} & \swarrow & V_{r,-s} & \leftarrow & V_{r,s+2p} & \leftarrow & V_{r,-s-2p} & \leftarrow & V_{r,s+4p} & \cdots \\ V_{r,s} & & & \times & & \times & & \times & & \\ & \nwarrow & V_{r,2p-s} & \leftarrow & V_{r,s-2p} & \leftarrow & V_{r,4p-s} & \leftarrow & V_{r,s-4p} & \cdots \end{array} \tag{1.2.24}$$

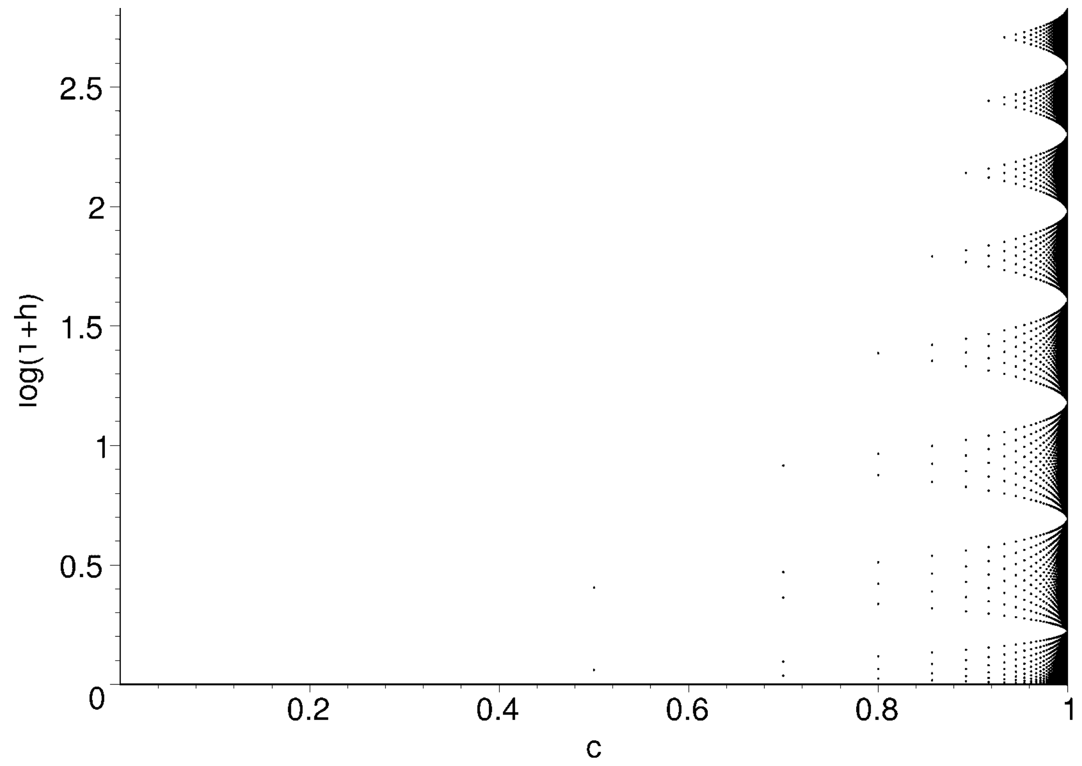



**Figure 1.4.:** The unitary Virasoro representations in a section of the $(h, c)$ plane

**Logarithmic Models ($p > p' = 1$):** The highest weight representations for the weights $h_{r,1} = \frac{(pr-1)^2-(p-1)^2}{4p}$ have the embedding structure

$$V_{1,s} \leftarrow V_{1,-s} \leftarrow V_{1,s-2p} \leftarrow V_{1,-s-2p} \leftarrow V_{1,s-4p} \ \ldots \tag{1.2.25}$$

Because of $h_{r+1,s} = h_{1,s-rp}$, $M_{r+1,s} \simeq M_{1,s-rp}$, the embedding structure for all other highest weight representations is fixed as well.

**Gaussian models ($p = p'$):** The embedding structure for all highest weight representations is given by $V_{r,s} \leftarrow V_{r,-s}$ with central charge $c = 1$.

**Case 2 ($k \in \mathbb{Q}$, $k' \in \mathbb{C} \setminus \mathbb{Q}$):** These so-called *parabolic models* (cf. [Flo93]) correspond to those $c \in \mathbb{Q}$ not included in the above discrete series. The set of rational weights consists of $h_{r,\pm r} \in \mathbb{Q}\ \forall r \in \mathbb{Z}$. The embedding structure for all highest weight representations is $V_{r,s} \leftarrow V_{r,-s}$.

**Case 3 ($k \in \mathbb{C} \setminus \mathbb{Q}$):** The central charge $c$ is irrational as well as all the $h_{r,s}$ except $h_{1,1} = 0$. The embedding structure for all highest weight representations is also $V_{r,s} \leftarrow V_{r,-s}$. These are *irrational models*.

### 1.2.6. Jordan Highest Weight Modules

As mentioned in section 1.2.2, it may occur that $L_0$ is not diagonalizable, a situation encountered e.g. in logarithmic conformal field theories, which are also discussed in this thesis.

There appear indecomposable representations. One is in need of a generalized concept. Instead of on a unique highest weight state, the representation is then built on an in general multi-dimensional so-called *Jordan cell*. Victor Gurarie is acknowledged to be the first who pointed out [Gur93] that if a theory contains two operators with the same conformal weight $h$ and in general the same set of quantum numbers with respect to the maximally extended chiral symmetry algebra, then logarithmic correlation functions appear and $L_0$ is no longer diagonalizable. More precisely, logarithmic operators necessarily appear in a conformal field theory whenever the differential equations arising from the conformal Ward identities and the existence of singular vectors yield degenerate solutions. The representation theory of the Virasoro algebra is then more complicated. The necessary basics are provided in this section. For a more detailed presentation, the reader is referred to [Flo03, Gab03a].

Suppose there exist $k$ operators $\Phi^{(n)}(z)$ with the same conformal weight $h$, $0 \leq n \leq k-1$, corresponding to $k$ linear independent states $|h^{(n)}\rangle$, forming a Jordan cell of rank $k$. An $\mathcal{V}$-module $M$ is called *Jordan highest weight module* of rank $k$ if the following requirements are fulfilled:

(i) $C|v\rangle = c|v\rangle \quad \forall |v\rangle \in M$

(ii) $L_0|h^{(0)}\rangle = h|h^{(0)}\rangle$ and $L_0|h^{(n)}\rangle = h|h^{(n)}\rangle + |h^{(n-1)}\rangle \quad \forall n \in \{1, \ldots, k-1\}$

(iii) $L_n|h^{(m)}\rangle = 0 \quad \forall n \in \mathbb{Z}_{\geq -1},\ m \in \{0, \ldots, k-1\}$

(iv) $M = \mathcal{U}|h^{(k-1)}\rangle$ .

Then, $h$ is called *highest weight* of the module and the $|h^{(n)}\rangle$ are called its *highest weight vectors*. In the case of $k = 2$, $|h^{(1)}\rangle$ is called *upper* and $|h^{(0)}\rangle$ is called *lower* highest weight vector. The fields $\Phi^{(n)}(z)$, $1 \leq n \leq k-1$, are called *logarithmic partner fields*, or just *logarithmic partners*, of the primary field $\Phi^{(0)}(z)$. Thus, the dimension of the Jordan cell is the rank $k$ of the indecomposable representation.

In section 1.4.1, the extension of the Virasoro symmetry algebra by chiral primary fields is described. Of course, it may also happen that the zero modes of these other generators are non-diagonalizable. In this thesis, however, only the representation theory of logarithmic conformal field theories with respect to the Virasoro algebra alone will be necessary.

Again, there is an abstract entity which can be mapped onto any Jordan highest weight module: A *Jordan Verma module* of highest weight $h$ is a Jordan highest weight module $V$ with highest weight $h$ that has the following universal property: For every Jordan highest weight module $M$ with highest weight $h$, there exists a unique (up to scalar multiples) $\mathcal{V}$-epimorphism $\Phi : V \to M$. The Jordan Verma module of highest weight $h$ is unique for all $h \in \mathbb{C}$.

Let $V$ be the Jordan Verma module with highest weight $h$ and highest weight vectors $|h^{(0)}\rangle$ and $|h^{(1)}\rangle$. Then, a base of $V$ is given by

$$\{L_{k_n} \ldots L_{k_1}|h^{(j)}\rangle \mid n \in \mathbb{Z}_{\geq 0}, j \in \{0,1\}, 0 > k_1 \geq \ldots \geq k_n\} \ . \qquad (1.2.26)$$

# 1.3. Minimal Models of the Virasoro Algebra

## 1.3.1. Operator Content of Minimal Models

As discussed in section 1.2.5, the conformal field theory models with central charge

$$c_{p,p'} = 1 - 6\frac{(p-p')^2}{pp'} \quad p, q \in \mathbb{Z}_{>1} \ , \tag{1.3.1}$$

with $p$ and $p'$ coprime, are called *minimal models*. In [BPZ84], it was pointed out that these models only possess a finite number of primary fields and that the operator algebra closes under fusion. All conformal weights of the primary fields in the model with central charge $c_{p,p'}$, wich will be denoted by $\mathcal{M}(p,p')$, are determined by

$$h^{p,p'}_{r,s} = \frac{(pr-p's)^2-(p-p')^2}{4pp'} \tag{1.3.2}$$

with $1 \leq r < p'$ and $1 \leq s < p$ and w.l.o.g. $p > p'$. The resulting table of possible weights is called *Kač table* or *conformal grid*.

## 1.3.2. The Unitary Series of Minimal Models

If the minimal model of the Virasoro algebra is demanded to be *unitary*, i.e. the sesquilinear form $\langle .,. \rangle$ is positive definite, then, as discussed in section 1.2.5, the set of possible central charges is restricted to

$$c_{p+1,p} = 1 - \frac{6}{p(p+1)} \ . \tag{1.3.3}$$

The corresponding distinct highest weights are then given by

$$h^{p+1,p}_{r,s} = \frac{((p+1)r-ps)^2-1}{4p(p+1)} \quad \text{with} \quad 1 \leq r \leq p-1,\ 1 \leq s \leq r \ . \tag{1.3.4}$$

To illustrate the class of minimal models a bit, let us give an example for a physical application of a certain unitary minimal model in the next section.

## 1.3.3. The Ising Model as a Minimal Model

The Ising model is about the simplest imaginable system in which large numbers of particles might be expected to produce some kind of cooperative behavior. In 1920, Wilhelm Lenz proposed the Ising model as a simplified version of a ferromagnet. Shortly after, Ernst Ising studied that model during his doctoral studies. It just consists of up and down spins arranged on a lattice. For neighboring spins, it takes less energy to point into the same direction than in opposite directions. But there is just this short-distance interaction, no lang-range interaction, and thus the spins are sort of randomly distributed by thermal fluctuations on the long range. This is indeed a very simplified model, the spins representing electrons, the lattice representing the crystalline structure of the ferromagnet and the nearest neighbor interaction representing the overlap of wave functions. If the temperature is high, thermal fluctuations completely randomize the spin orientation. If the temperature is zero, the spins are all aligned. In a one-dimensional Ising model, it turns out there is no phase transition, but in a two-dimensional Ising model, there is an abrupt magnetization (number of up spins minus number of down spins) at a certain temperature, which corresponds to a second order phase transition at the so-called *critical point* between a disordered phase at high temperature and an ordered phase at low temperature, just as experiments have shown it to be the case. The Boltzmann weight $e^{-\frac{E}{kT}}$ divided by the partition function $Z$ determines the probability of a certain state depending on temperature and energy. Onsager in 1944 solved the problem of computing the partition function for the two-dimensional Ising model in the thermodynamic limit (as the number of spins tends to infinity) and was able to predict the critical temperature. The magnetization varies with the temperature according to the power law $M \sim (T_C - T)^{\frac{1}{8}}$ in its ferromagnetic phase, and for $T > T_C$, there is no magnetization. This defines the *critical exponent* $\beta = \frac{1}{8}$. The behavior of thermodynamic functions near or at the critical point is characterized by critical exponents defining power laws as $T$ tends towards $T_C$. There are six common exponents of this type for the Ising model, which can all be expressed in terms of two of them, $\eta$ and $\nu$. In two dimensions, the exact solution yields $\eta = \frac{1}{4}$ and $\nu = 1$.

The unitary minimal model $\mathcal{M}(4,3)$, which has the smallest non-trivial Kač table of all unitary minimal models[7], describes the *critical Ising model*. It consists of the Ising spin $\sigma$ and the energy density $\varepsilon$, both of them being continuum versions of the lattice spin $\sigma_i$ and the interaction energy $\sigma_i\sigma_{i+1}$, respectively. The two-point correlation functions in conformal field theory display the critical exponents $\eta$ and $\nu$. In two dimensions,

$$\langle \sigma_i \sigma_{i+n} \rangle = \frac{1}{|n|^{\eta}} \tag{1.3.5}$$

$$\langle \varepsilon_i \varepsilon_{i+n} \rangle = \frac{1}{|n|^{4-\frac{2}{\nu}}} \ . \tag{1.3.6}$$

Assuming that the scaling fields $\sigma$ and $\varepsilon$ have no spin, i.e. $h = \bar{h}$, (1.3.5) and (1.3.6) imply via (1.1.34) that their conformal dimensions are

$$h_\sigma = \bar{h}_\sigma = \frac{1}{16} \quad \text{and} \quad h_\varepsilon = \bar{h}_\varepsilon = \frac{1}{2} \ . \tag{1.3.7}$$

Together with the identity operator we have thus three operators in the holomorphic part of the theory, whose weights can all be found in the Kač table displayed in figure 1.5 of the minimal conformal field theory model $\mathcal{M}(4,3)$, which has the smallest possible non-trivial value of the effective central charge $c_{\text{eff}}^{4,3} = c_{4,3} = \frac{1}{2}$. Thus, the minimal model $\mathcal{M}(4,3)$ can be identified [BPZ84] with the critical Ising model, the fusion rules (cf. section 3.4.2) being

$$\sigma \times \sigma = \mathbb{1} + \varepsilon \tag{1.3.8}$$

$$\sigma \times \varepsilon = \sigma \tag{1.3.9}$$

$$\varepsilon \times \varepsilon = \mathbb{1} \ , \tag{1.3.10}$$

compatible with the $\mathbb{Z}_2$ symmetry $\sigma_i \mapsto -\sigma_i$ of the Ising model. Note also that there is an equivalence to the free Majorana fermion model with $c = \frac{1}{2}$.

[7]The non-unitary minimal model $\mathcal{M}(5,2)$ has an even smaller Kač table, corresponding to the Yang-Lee model and the trivial $\mathcal{M}(3,2)$ model has just central charge $c = 0$, $\mathcal{H} = \mathbb{C}$ and thus the single character $\chi = 1$.

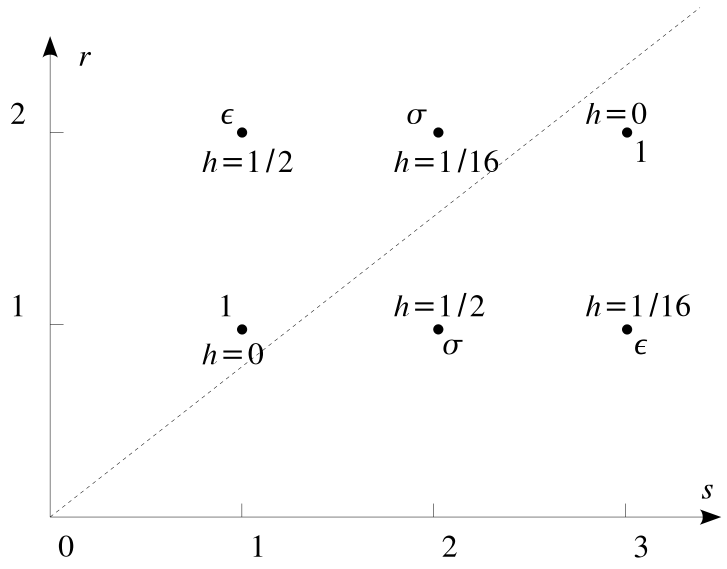



**Figure 1.5.:** The Kač table for the $\mathcal{M}(4,3)$ minimal model which corresponds to the critical Ising model

## 1.4. $\mathcal{W}$-Symmetry

### 1.4.1. Extending the Symmetry Algebra

The Virasoro algebra, which is always present in a conformal field theory, only leads to rational models for values of the central charge that are smaller than one. These are the minimal models with $c_{p,p'} = 1-6\frac{(p-p')^2}{pp'}$. Even the effective central charge $c_{\text{eff}}^{p,p'} = c_{p,p'} - 24h_{\min}$ has to be smaller than one in this case. For the unitary series of minimal models, $c_{\text{eff}} = 1 - \frac{6}{pp'}$. However, some applications concerning second order phase transitions are known which also need rational conformal field theories with $c > 1$, e.g. the two-dimensional spin models with $k$ spin states, the so-called $\mathbb{Z}_k$ *parafermions*. In [FZ85], these were shown by Vladimir A. Fateev and Zamolodchikov to have conformal field theories with central charges $c = \frac{2k-2}{k+2}$ describing their second order phase transition. Moreover, important applications in string theory require $N = 2$ supersymmetric rational conformal field theories with $c = 9$ [Gep87, Gep88]. But these theories are not rational with respect to the Virasoro algebra. If the conformal field theory is invariant under an extension of the conformal algebra and one enlarges the symmetry algebra in this manner, the theory still might be rational with respect to that enlarged symmetry algebra. For example, degeneracies in the spectrum of conformal dimensions can be resolved by the quantum numbers that correspond to the additional symmetry. Alexander B. Zamolodchikov [Zam85] is respected to be the first who studied the possibility of adding another primary field to the theory. He thus enlarged the symmetry algebra by the modes of the additional primary field. These enlarged symmetry algebras were thenceforward called $\mathcal{W}$-algebras. In general, the keynote to this is that additional field operators are introduced whose modes map between different representation modules $M(c,h)$ and by doing so, they rearrange several conformal families into a single one. When rearranging infinitely many conformal families into a single family, one may hope to describe the field theory with finitely many of these bigger families, which results in the theory being rational with respect to the $\mathcal{W}$-algebra. The original approach used chirally non-local fields. However, one would like to use better behaved chiral fields for such a mapping: The primary local chiral fields.

A local chiral primary field $\Phi(z) = \sum_{n-h_\Phi \in \mathbb{Z}} \Phi_n z^{-n-h_\Phi}$ of conformal dimension $h_\Phi \in \frac{\mathbb{Z}_{\geq 0}}{2}$ is characterized by the commutator of its modes with the Virasoro algebra

$$[L_n, \Phi_m] = (n(h_\Phi - 1) - m)\Phi_{n+m} \ , \tag{1.4.1}$$

which implies for the vacuum vector (cf. section 1.2.1)

$$\Phi_n|0\rangle = 0 \quad \forall n > -h_\Phi \tag{1.4.2}$$

$$\Phi_{-h_\Phi}|0\rangle = |h_\Phi\rangle \ . \tag{1.4.3}$$

The commutator (1.4.1) does not close in primary fields only but also produces quasi-primary fields, which are defined in that they only need to satisfy (1.4.1) for $m \in \{-1, 0, 1\}$, i.e. the global conformal group. There is the following general formula for the modes of two quasi-primary local chiral fields $\Phi^i$ and $\Phi^j$ [Nah89]:

$$[\Phi^i_n, \Phi^j_m] = d_{ij}\delta_{n+m,0}\binom{n + h_{\Phi^i} - 1}{2h_{\Phi^i} - 1} + \sum_{\substack{k \in I \\ h_{\Phi^k} < h_{\Phi^i} + h_{\Phi^j}}} C^k_{ij} p_{ijk}(m, n)\Phi^k_{m+n} \ . \tag{1.4.4}$$

Note that the Virasoro algebra is contained in this commutator formula with $h_T = 2$. The structure constants $d_{ij}$ and $C_{ijk} = \sum_l d_{kl} C^l_{ij}$ can be expressed in terms of the two- and three-point functions

$$C_{ijk} = \langle 0|\Phi^k_{h_{\Phi^k}} \Phi^i_{-h_{\Phi^k}+h_{\Phi^j}} \Phi^j_{-h_{\Phi^j}}|0\rangle \tag{1.4.5}$$

$$d_{ij} = \langle 0|\Phi^i_{h_{\Phi^i}} \Phi^j_{-h_{\Phi^j}}|0\rangle \tag{1.4.6}$$

and the *universal polynomials* are given by

$$p_{ijk}(m, n) = \sum_{r,s \in \mathbb{N}} \delta_{r+s, h_{ijk}-1} a^r_{ijk} \binom{m + n - h_{\Phi^k}}{r} \binom{h_{\Phi^i} - m - 1}{s} \tag{1.4.7}$$

with

$$a^r_{ijk} = \binom{2h_{\Phi^k} + r - 1}{r}^{-1} \binom{h_{\Phi^i} + h_{\Phi^k} - h_{\Phi^j} + r - 1}{r} \ . \tag{1.4.8}$$

Furthermore, it is necessary to add a correction term to the usual normal ordered product $N(\Phi^i, \partial^n \Phi^j)$ to be also valid for quasi-primary fields. The quasi-primary normal ordering prescription $\mathcal{N}(\Phi^i, \partial^n \Phi^j)$ then produces a field of dimension $h_{\Phi^i} + h_{\Phi^j} + n$.

Consequently, since rational conformal field theories with $c > 1$ can be constructed using a $\mathcal{W}$-symmetry algebra bigger than just the Virasoro algebra, it is to be expected that these $\mathcal{W}$-algebras are an important ingredient for the task of classification of all rational conformal field theories. The presence of extra symmetry can make it possible to have a finite decomposition of the Hilbert space of physical states in terms of irreducible representations of the extended algebra and thus to form rational conformal field theories.

A definition of what will be understood of a $\mathcal{W}$-algebra is in order. One no longer has a Lagrangian approach to the theory but instead defines it from its correlation functions, the latter being the fundamental objects of a (chiral) conformal field theory. Given a complete set of amplitudes, all data of a chiral conformal field theory can be reconstructed. This is also the case for non-chiral theories [GK99, GG00].

A *meromorphic conformal field theory* [God89, BS93] is a quadruple $(\mathcal{F}, \mathcal{H}, \mathbb{V}, T)$ which consists of a characteristic Hilbert space of states $\mathcal{H}$ (referred to as the vacuum module), a dense subspace $\mathcal{F}$, which is the Fock space of states of *finite* occupation number, a map $|\Phi\rangle \to \Phi(z)$, the so-called *vertex operator map*, from $\mathcal{H}$ into the space of fields, and furthermore a distinguished state $|T\rangle$ whose corresponding field $T(z)$ is the traceless stress-energy tensor of the theory. The modes $L_n$, $n \in \mathbb{Z}$, of the stress-energy tensor form the Virasoro algebra. The image of the vertex operator map, i.e. the space of fields, has to be a *local system* $\mathbb{V}$. Thus, the following conditions have to be fulfilled:

(i) There exists a unique state $|0\rangle \in \mathcal{F}$, the vacuum, such that $\Phi(z)|0\rangle = \mathrm{e}^{zL_{-1}}|\Phi\rangle$.

(ii) $\langle\Phi_1|\Phi(z)|\Phi_2\rangle$ is a meromorphic function of z.

(iii) $\langle\Phi_1|\Phi(z)\chi(w)|\Phi_2\rangle$ is a meromorphic function for $|z| > |w|$.

(iv) $\langle\Phi_1|\Phi(z)\chi(w)|\Phi_2\rangle = \epsilon_{\Phi\chi}\langle\Phi_1|\chi(w)\Phi(z)|\Phi_2\rangle$ by analytic continuation, where $\epsilon_{\Phi\chi} = -1$ if both $\Phi$ and $\chi$ are fermionic and $\epsilon_{\Phi\chi} = 1$ otherwise.

Note that a function which is analytic everywhere in the finite plane (i.e. everywhere except at $\infty$) except at a finite number of poles is called a *meromorphic function*.

A consequence of these axioms is that there is a one-to-one correspondence between the states and the fields (cf. 1.2.1), i.e. the vertex operator map is an isomorphism. The locality assumption also implies that only fields with integer (bosonic) or half-odd-integer (fermionic) conformal weights $h$ can occur in a meromorphic conformal field theory, since otherwise the two-point function would have branch cuts.

A $\mathcal{W}$*-algebra* is a meromorphic conformal field theory [God89] generated by a finite number of distinguished *simple* (i.e. not a normal ordered product of other fields) quasi-primary fields $\Phi^1, \Phi^2, \ldots, \Phi^k$ of conformal dimensions $h_{\Phi^1}, h_{\Phi^2}, \ldots, h_{\Phi^k}$, respectively, where $\Phi^1$ is the Virasoro stress-energy tensor. This algebra is then called $\mathcal{W}(2, h_{\Phi^2}, \ldots, h_{\Phi^k})$, where $\mathcal{W}(2)$ denotes the Virasoro algebra. $\mathcal{W}$-algebras are higher-spin bosonic extensions of the Virasoro algebra. It can be shown from the definition that $\mathcal{F}$ is spanned by lexicographically ordered states

$$\Phi^{i_1}_{-m_1-2}\Phi^{i_2}_{-m_2-h_{\Phi^2}}\cdots\Phi^{i_n}_{-m_n-h_{\Phi^n}}|0\rangle \tag{1.4.9}$$

where $h_{\Phi^{i_j}} \geq h_{\Phi^{i_{j+1}}}$, $i_j = i_{j+1} \Rightarrow m_j \geq m_{j+1}$ and $m_j \geq 0$. Conversely, if $\mathcal{F}$ is spanned by states of this form then all the fields can be written as normal ordered products of the $\Phi^i(z)$ and their derivatives. As has been proven in [EFH$^+$92] by an analysis of character asymptotics, $\mathcal{W}$-algebra models can only be rational if the effective central charge is smaller than the theory's number of bosonic generators plus half its number of generating fermionic fields. In the Virasoro case, this reduces to just the stress-energy tensor, which confirms that Virasoro algebra models can only be rational if $c_{\text{eff}} < 1$.

In general, a rational meromorphic conformal field theory is a chiral algebra for which Zhu's algebra is finite-dimensional [Zhu96]. However, there are also theories which are non-chiral, e.g. the series of triplet algebras discussed in section 1.4.3, the latter being logarithmic theories that do not factorize into standard chiral theories. Non-meromorphic operator products will become important then in order to construct a local logarithmic conformal field theory [CF06, GG00, GK99]. Further information about $\mathcal{W}$-symmetry may be found in [BS93] and in the collection of reprints [BS95].

### 1.4.2. $\mathcal{W}$-Algebras of Central Charge $c = 1 - 24k$

Case 2 from the classification of all possible structures of degenerate highest weight representations of the Virasoro algebra concerns the so-called parabolic models. The Hilbert space of these theories decomposes into infinitely many Virasoro highest weight representations. Thus, these theories are not rational with respect to the Virasoro algebra. However, one can enlarge the theory's underlying symmetry algebra and construct certain $\mathcal{W}$-algebras, with respect to which the Hilbert space decomposes into finitely many highest weight representations. Therefore, the theory then is rational with respect to this enlarged symmetry algebra. In [BFK$^+$90], $\mathcal{W}$-algebras with one or two additional generators were constructed which exist only for finitely many discrete values of the central charge. Among these discrete values are two series of central charges, namely for a given $\delta$ either $c = 1 - 8\delta$ or $c = 1 - 3\delta$. In these cases, the additional primary field has conformal dimension $\delta = h_{2,2}$ or $\delta = h_{3,3}$, and thus the two series correspond to the algebras $\mathcal{W}(2, \delta)$. From the Kač determinant formula, one can read off that these fields belong to degenerate Virasoro highest weight representations. Since $c$ is rational and since $c \neq 1 - 6\frac{(p-p')^2}{pp'}$ for some coprime $p$ and $p'$, these theories have to be of the parabolic type, i.e. case two. To obtain a rational conformal field theory, it is necessary to regroup the infinitely many Virasoro highest weight representations into finitely many $\mathcal{W}$-algebra representations. This is only possible [Fel89] with a non-trivial, local system of chiral vertex operators, which implies that the difference between any two of the primary fields has to be integer spaced. Local chiral operators then map between the corresponding states [God89].

(1.2.23) implies that the pairwise difference between all of the highest weights $h_{r,s}(k)$ is integer if $s = \pm r$ and $k$ integer or half-integer (or $k \in \frac{\mathbb{Z}_{\geq 0}}{4}$ for $r$ odd). Then, in particular,

$$h_{r,r} = (r^2 - 1)k \quad \text{and} \quad h_{r,-r} = (r^2 - 1)k + r^2 \tag{1.4.10}$$

either integer for all $r$ if $k$ integer or half-integer for all $r$ if $k$ half-integer. It is not possible to form a local system of chiral vertex operators for other values of $k$ [Flo93]. This local system of chiral fields is generated by normal ordered products of (derivations of) the finitely many simple primary fields, i.e. the field corresponding to $h_{2,2} = 3k$, in which case we get a $\mathcal{W}(2, 3k)$, or the field $h_{3,3} = 8k$, in case of which we get a $\mathcal{W}(2, 8k)$. (A field is called simple if it is not the normal ordered product of two other fields.)[8]

[8]The field corresponding to $h_{3,3}$ is not simple when the field corresponding to $h_{2,2}$ is also present in the theory, since the former is the primary projection of the normal ordered product of the latter with its $2k$th derivative. If one allows also non-simple fields to generate $\mathcal{W}$-algebras, then one may also construct a $\mathcal{W}(2, 3k, 8k)$.

### 1.4.3. Triplet $\mathcal{W}$-Algebras

This infinite series of logarithmic conformal field theories is generated by the Virasoro stress-energy tensor $T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, which is responsible for the conformal symmetry, and by an $SO(3)$-triplet of primary fields $W^a(z) = \sum_{n\in\mathbb{Z}} W^a_n z^{-n-(2p-1)}$ with $a \in \{-1, 0, 1\}$. These algebras are called $\mathcal{W}(2, 2p-1, 2p-1, 2p-1)$, or also *triplet algebras*, and a triplet algebra exists for any $p \geq 2$ [Kau91]. As noted in chapter 1.3, for the values $c_{p,p'} = 1 - 6\frac{(p-p')^2}{pp'}$ of the central charge, one has the series of minimal models, which are rational with respect to even the Virasoro algebra only. It was noticed by Horst G. Kausch [Kau91] that the $\mathcal{W}(2, 2p-1, 2p-1, 2p-1)$ algebras have central charges $c_{p,1} = 1 - 6\frac{(p-1)^2}{p}$. But these are not Virasoro-minimal models, since for Virasoro minimal models, $p$ and $p'$ both have to be greater than one and coprime. However, the modular transformation $\mathcal{S}$ has a finite-dimensional representation on the space of generalized character functions, as will be described below. While the situation that $\mathcal{S}$ has a finite-dimensional representation on the space of the characters is regarded as the definition of rational conformal field theories, one cannot directly apply this terminology here, since one only has generalized character functions including logarithmic terms. This stems from the fact that the triplet algebra does not consist of irreducible modules alone, but also of reducible but indecomposable representations, because it has more than one generator with the same $h$-value. Keeping this situation in mind, one may still call the triplet algebra models rational. In [CF06], the equivalence between so-called $C_2$-*cofiniteness* and this generalized form of rationality is conjectured, which expands the former conjecture by Dong and Mason [DM96], which turned out to be wrong. Dong & Mason's conjecture stated that $C_2$-cofiniteness should be equivalent to a theory having finitely many representations that are completely decomposable, i.e. that are rational in the strong mathematical sense.

Let us now consider the case of $p = 2$, i.e. $c = -2$, in more detail. This model plays an important role in the treatment of two-dimensional polymers and self-avoiding walks [DS87]. Its commutors can be calculated from (1.4.4) as

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n^2(n-1)\delta_{n+m,0} \tag{1.4.11}$$

$$[L_n, W^a_m] = (2n-m)W^a_{n+m} \tag{1.4.12}$$

$$[W^a_n, W^b_m] = g^{ab}\Bigg(2(n-m)\Lambda_{n+m} + \frac{1}{20}(n-m)(2n^2+2m^2-nm-8)L_{n+m} \tag{1.4.13}$$

$$- \frac{1}{120}n(n^2-1)(n^2-4)\delta_{n+m,0}\Bigg) \tag{1.4.14}$$

$$+ f^{ab}_c\left(\frac{5}{14}(2n^2+2m^2-3nm-4)W^c_{n+m} + \frac{12}{15}V^c_{n+m}\right) \tag{1.4.15}$$

with

$$\Lambda = \mathcal{N}(T,T) = N^{(2)}(T,T) - \frac{3}{10}\partial^2 T \tag{1.4.16}$$

$$V^a = \mathcal{N}(W^a,T) = N^{(2)}(W^a,T) - \frac{3}{14}\partial^2 W^a\ , \tag{1.4.17}$$

where $g^{ab}$ and $f_c^{ab}$ are the metric and the structure constants of $su(2)$, respectively. If an orthonormal basis is chosen, then $g^{ab} = \delta^{ab}$ and $f_c^{ab} = \mathrm{i}\varepsilon^{abc}$. It is also possible to calculate the coefficients for $p > 2$, but the resulting expressions would become too messy to be printed here. But which representations actually are allowed in the triplet algebra case at $p = 2$? The triplet algebra at $c = -2$ is associative because certain states in the vacuum representation are null that would otherwise violate associativity. Matthias Gaberdiel and Horst Kausch [GK96] noted that the number of highest weight modules reduces from infinitely many in the Virasoro case to four in the triplet algebra case, though in the latter case, these are regarded as generalized highest weight modules. This comes about as follows: The keynote is that if one has a vacuum representation and knows its null-vectors, then this puts restrictions on the other representations: The latter have to be *compatible* with the vacuum representation, meaning that any correlation function of any fields corresponding to states of any representation, which also include the field corresponding to the null vector of the vacuum representation, has to vanish. Otherwise, the structure of the vacuum representation would have been modified by the other representations (see [GG00, Gab03a]). This implies in particular that the zero modes of the null-states have to vanish on the highest weight states of all representations. The action on those highest weight states can be expressed through the fields in the theory. In the case of the $c = -2$ triplet algebra, this leads to the constraint

$$\left(W_0^a W_0^b - g^{ab}\frac{1}{9}L_0^2(8L_0+1) - f_c^{ab}\frac{1}{5}(6L_0-1)W_0^c\right)|h\rangle = 0\ , \tag{1.4.18}$$

where $|h\rangle$ is any highest weight state. From (1.4.18), it follows that

$$L_0^2(8L_0+1)(8L_0-3)(L_0-1)|h\rangle = 0\ . \tag{1.4.19}$$

This, in turn, implies that $L_0$ has to take the values $h \in \{-\frac{1}{8}, 0, \frac{3}{8}, 1\}$ on the highest weight states. Four irreducible representations are allowed. But additionally, since it is only required that $L_0^2 = 0$, also a two-dimensional space of highest weight states is allowed:

$$L_0\omega = \Omega\ , \quad L_0\Omega = 0\ . \tag{1.4.20}$$

This corresponds to a reducible but indecomposable representation. In fact, there is even another indecomposable representation which can not be detected by (1.4.19) since it is not generated by a highest weight state. The presence of reducible but indecomposable representations implies a Jordan cell structure (see section 1.2.6) in $L_0$, i.e. $L_0$ is not diagonalizable but necessarily has off-diagonal components. Since this finite set of highest weight representations closes under fusion, the triplet algebra was called rational in [GK96]. Condition (1.4.18) leads to the algebra of zero modes

$$[W_0^a, W_0^b] = \frac{2}{5}(6h-1)f_c^{ab}W_0^c\ . \tag{1.4.21}$$

The generators can be rescaled to get the well-known $su(2)$ algebra. Because of that, just as in the $su(2)$ case, we can label the irreducible representations of the zero mode algebra (i.e. of $su(2)$) by $j$ and $m$. The Casimir operator $\sum_{a=1}^{3}(W_0^a)^2$ then takes the value $j(j+1)$ on the highest weight states and $W_0^3$ takes the value $m$. The operators which change the value

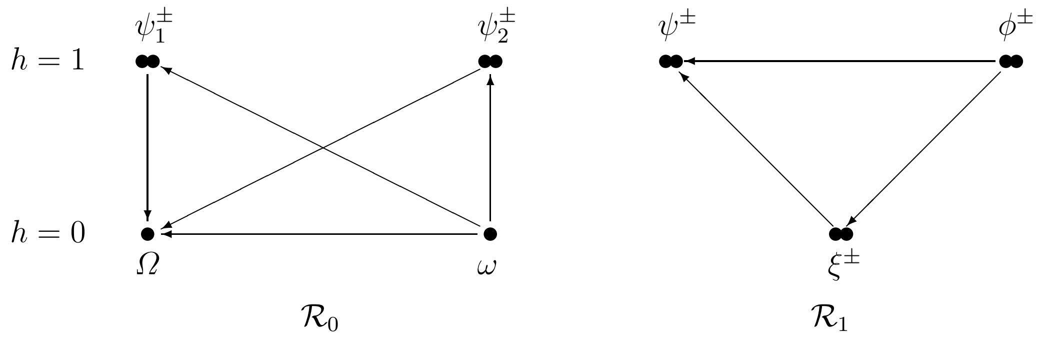



**Figure 1.6.:** The two indecomposable representations of the triplet algebra at $c = -2$.

of $m$ are then $W_0^\pm = W_0^1 \pm \mathrm{i}W_0^2$. If we set $a = b$ in (1.4.18), it follows that $(W_0^a)^2$ acts the same for all $a \in \{1, 2, 3\}$ on the highest weight states, and thus

$$\sum_{a=1}^{3}(W_0^a)^2|h\rangle = j(j+1)|h\rangle = 3m^2|h\rangle \ , \quad \text{i.e. } j(j+1) = 3m^2 \ . \tag{1.4.22}$$

The only solutions to this are $j \in \{0, \frac{1}{2}\}$. The result is that there are four irreducible representations, namely for $j = 0$ there are singlet representations $\mathcal{V}_h$ with $h \in \{0, -\frac{1}{8}\}$ and for $j = \frac{1}{2}$ there are doublet representations $\mathcal{V}_h$ with $h \in \{\frac{3}{8}, 1\}$ [EHH93, GK96]. Note that a more mathematically detailed analysis can be carried out and involves the concept of Zhu's algebra [Zhu96]. Formally, these highest weights can be read off the *extended Kač table* for the 'augmented minimal model' [Flo97, EF06] $\mathcal{M}(3p, 3)$ which, of course, is not a minimal model since $3p$ and 3 are not coprime.

When calculating the fusion products of these four irreducible representations, one finds that the fusion algebra does not close, but that instead the two indecomposable representations $\mathcal{R}_0$ and $\mathcal{R}_1$ have to be included [GK96]. The latter two can be described by the diagram 1.6, where each vertex denotes an irreducible representation $\mathcal{V}_0$ or $\mathcal{V}_1$ corresponding to the highest weight vector written next to it. An arrow indicates the action of the triplet algebra.

For example, the module with $h = 1$ has an irreducible submodule generated by a vector $\psi$, but also another vector $\phi$ which does not belong to this submodule and which has the property that

$$L_0\phi = \phi + \psi \ . \tag{1.4.23}$$

In particular, $\mathcal{R}_0$ has the defining relations

$$L_0\omega = \Omega \qquad\qquad W_0^a\omega = 0 \tag{1.4.24a}$$
$$L_0\Omega = 0 \qquad\qquad W_0^a\Omega = 0 \tag{1.4.24b}$$
$$L_n\omega = 0 \quad \forall n > 0 \qquad\qquad W_n^a\omega = 0 \quad \forall n > 0 \ , \tag{1.4.24c}$$

where $\omega$ is the highest weight vector of $\mathcal{R}_0$, forming a Jordan cell with the vacuum vector $\Omega$. Both have highest weight $h = 0$. $\mathcal{R}_0$ includes the vacuum irreducible representation $\mathcal{V}_0$.

Meanwhile, the defining relations of $\mathcal{R}_1$ are

$$L_0\psi = \psi \qquad W_0^a\psi = 2t^a\psi \tag{1.4.25a}$$
$$L_0\xi = 0 \qquad W_0^a\xi = 0 \tag{1.4.25b}$$
$$L_{-1}\xi = \psi \qquad W_{-1}^a\xi = t^a\psi \tag{1.4.25c}$$
$$L_1\phi = -\xi \qquad W_1^a\phi = -t^a\xi \tag{1.4.25d}$$
$$L_0\phi = \phi + \psi \qquad W_0^a\phi = 2t^a\phi \ , \tag{1.4.25e}$$

with a doublet $\phi^\pm$ of generating states forming an $L_0$ Jordan cell with another doublet $\psi^\pm$, both corresponding to weight $h = 1$. However, note that $\phi^\pm$ are not highest weight states: The ground state doublet $\xi^\pm$ has $h = 0$. Thus, $\mathcal{R}_1$ is not a highest weight representation. All the described representations together close under fusion, hence the theory may be called rational (in the generalized sense discussed above).

# 2. Modular Invariance

## 2.1. The Modular Group $\Gamma$

Conformal field theories may in general be defined on arbitrary Riemann surfaces. It is a general result for a large class of conformal field theories that having crossing symmetry of correlators on the complex plane and modular invariance of the partition function on the torus is sufficient in constructing a consistent theory on arbitrary Riemann surfaces. A *Riemann surface* (see e.g. [Jos02]) is a two-dimensional manifold together with a conformal structure. A conformal structure is obtained by adding all compatible charts to a conformal atlas. An atlas is said to be conformal if all the transition maps between the charts are holomorphic.[1] Recall that on a Riemann surface, if a map is analytic and its first derivative does not vanish in a region $\mathcal{R}$, then the map is conformal in $\mathcal{R}$. Each Riemann surface has a *genus* $g$, which corresponds loosely speaking to its number of 'handles'. The conformal field theory defined on that Riemann surface is then invariant under the *Fuchsian group* corresponding to genus $g$, which in the string theory context corresponds to the number of loops of a given Feynman diagram of its low-energy effective theory, i.e. quantum field theory. Incoming and outgoing strings are described by tubes attached to the Riemann surface which are effectively just replaced by punctures. The simplest Riemann surfaces would be $\mathbb{C}$ or open subsets of $\mathbb{C}$. In the previous chapter, conformal field theories on the simplest possible worldsheet, the cylinder, were considered. The cylinder was mapped by a conformal transformation to the punctured complex plane. The most important example of a compact Riemann surface is the Riemann sphere $S^2 \subset \mathbb{R}^3$ which can be mapped conformally to the extended complex plane $\mathbb{C} \cup \{\infty\}$. The second most simple example of a compact Riemann surface is the torus. If a conformal field theory is defined on a torus, i.e. periodic boundary conditions are imposed, then it is invariant under the *modular group* $\Gamma = PSL(2,\mathbb{Z})$, itself being a special case of the general Fuchsian group.

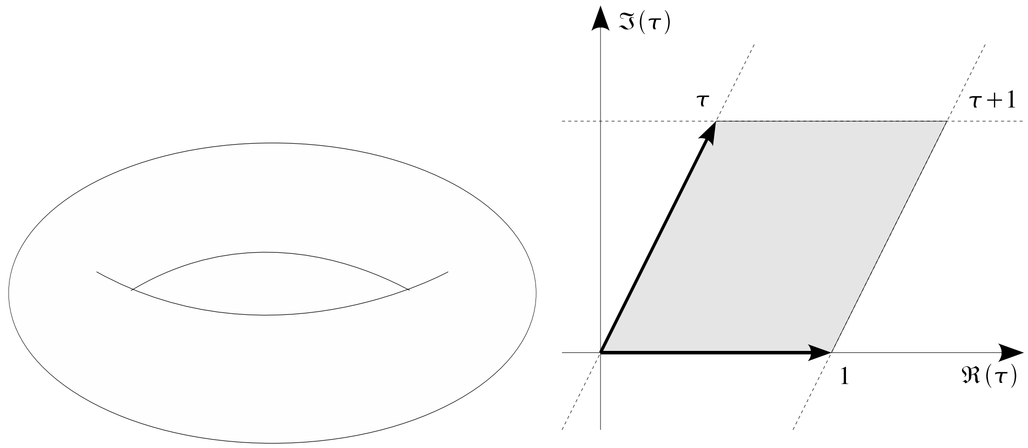
It can be useful to put a conformal field theory on a torus: Statistical systems of finite size with periodic boundary conditions are automatically defined on a torus if that area is a parallelogram. Or, if there is a puncture on opposite sides of the torus, then it could be the world-sheet of a closed string doing a one-loop process, just like a cylinder would be the world sheet of a freely moving closed string.

A torus is the set

$$\mathbb{T} = \mathbb{C}/L = \{z \mid z \simeq z + n\lambda_1 + m\lambda_2\} \tag{2.1.1}$$

where $L$ is the torus lattice and $\lambda_1$ and $\lambda_2$ are two linear independent lattice vectors, on the complex plane represented by two complex numbers, called the *periods* of the lattice. Thus, a torus is the complex plane modulo a lattice. It can be constructed by identifying points that differ only by a combination of lattice vectors: Roll up the area whose corners

[1] The term analytic is used synonymously with holomorphic in the physics literature.




(a) a torus $\mathbb{T}$ (b) the torus lattice $L$

**Figure 2.1.:** The torus

are at 0, $\lambda_1$, $\lambda_1 + \lambda_2$ and $\lambda_2$ to get a cylinder and then bend it and glue the two ends of the cylinder together to get a torus.[2] Or, equivalently, a torus is a Riemann sphere with a hole in it: Cut a circle around the north pole and a circle around the south pole and glue the two boundaries together.

Two-dimensional general coordinate invariance (see section 1.1.2) may be used to straighten the coordinate lines, translation invariance to fix one lattice point at the origin of the coordinate system, and rotation invariance of the theory allows to align one of the lattice vectors to the real axis. Furthermore, because of scale invariance of the conformal field theory, only the ratio

$$\tau = \frac{\lambda_1}{\lambda_2} \in \mathbb{H} \subset \mathbb{C} \tag{2.1.2}$$

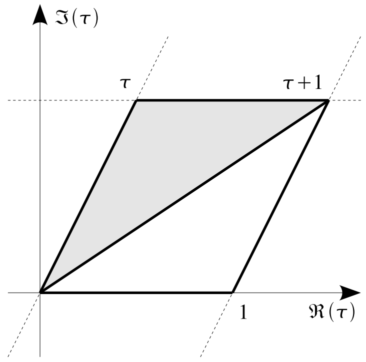
is needed to distinguish between inequivalent tori, where $\mathbb{H}$ is the upper half-plane. Thus, the two periods $\lambda_1$ and $\lambda_2$ are replaced simply by 1 and $\tau$, as displayed in figure 2.1. The entire lattice is described by just one complex number, which one conventionally chooses to be from the upper half-plane, i.e. $\Im(\tau) > 0$.

A torus has to be invariant under certain transformations of the lattice vectors: Obviously, choosing lattice vectors 1 and $\tau{+}1$ instead of 1 and $\tau$ describes the same lattice, as is depicted in figure 2.2. But there is another discrete symmetry that is not that obvious: One may also interchange the two lattice vectors and describe the other lattice vector in terms of $\tau$, as shown in figure 2.3. This can be achieved by first rotating the $\tau$ vector into the real axis and doing a rescaling by the transformation $\tau \mapsto \frac{1}{\tau}$ (figure 2.3(b)) and in the end choosing the new $\tau$ vector in the upper half-plane again which amounts to taking $-\frac{1}{\tau}$ (figure 2.3(c)).
These two discrete symmetries generate the modular group $\Gamma$ and are denoted

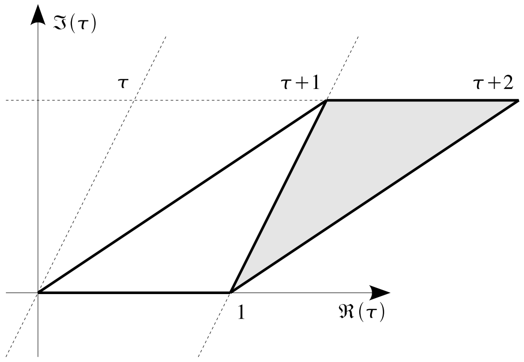
$$\mathcal{T} : \tau \to \tau + 1 \tag{2.1.3}$$

$$\mathcal{S} : \tau \to \ -\frac{1}{\tau}\ . \tag{2.1.4}$$

[2]Note by the way that if one of the ends of the cylinder is turned inside out and the two ends are then glued together (obviously not possible with a real tube), then one obtains a Klein bottle. You can play chess on a torus or a Klein bottle at http://www.geometrygames.org/TorusGames/index.html .




(a) The torus lattice




(b) The torus lattice after application of $\tau \mapsto \tau + 1$

**Figure 2.2.:** The modular transformation $\mathcal{T} : \tau \mapsto \tau + 1$

They may be represented by $2 \times 2$ matrices acting on $\left(\begin{smallmatrix}\lambda_1\\ \lambda_2\end{smallmatrix}\right)$ as

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \tag{2.1.5}$$

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \tag{2.1.6}$$

a translation and a reflection, respectively. Then, the general modular transformation is given by

$$\mathcal{U} : \tau \to \frac{a\tau + b}{c\tau + d} \quad \text{or in matrix form by} \tag{2.1.7}$$

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \tag{2.1.8}$$

where $a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$ (area preserving, i.e. guaranteeing that $U$ has an integer inverse) and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \simeq -\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ because an overall sign in the numerator and denominator of (2.1.7) cancels. Thus, $U \in PSL(2, \mathbb{Z})$. The two generators can easily be shown to satisfy the relations

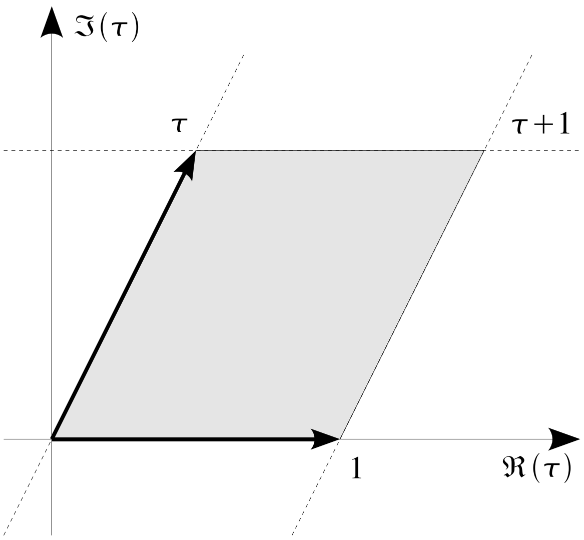
$$(\mathcal{S}\mathcal{T})^3 = \mathcal{S}^2 = \text{id} , \tag{2.1.9}$$

that is, every element from $PSL(2, \mathbb{Z})$ is a 'word' generated by $S$ and $T$ of the form

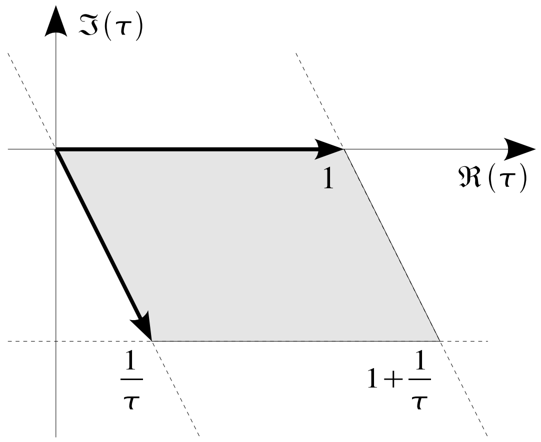
$$T^{n_0} S T^{n_1} S T^{n_2} \cdots \tag{2.1.10}$$

where $n_0 \in \{0, 1\}$, $n_i \geq 1$ and if $n_i = n_{i+1} = n_{i+2}$, then $n_i \neq 1$. Note that the modular group will keep $\tau$ on the upper half-plane.

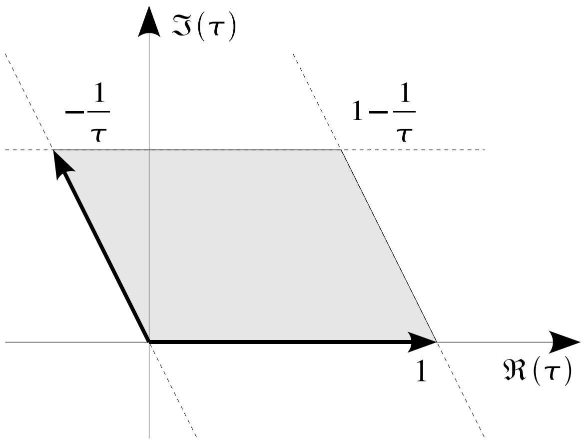
$n$-point correlation functions of conformal field theories on the torus imply representation spaces of the modular group. This is called modular covariance. The zero-point functions,which are the characters of the theory (i.e. propagators which have been cut short, the so-called *vacuum bubbles*), are especially interesting for us. If the characters form a finite-dimensional representation of the modular group, the conformal field theory is said to




(a) The torus lattice




(b) The torus lattice after application of a rotation and a rescaling by means of $\tau \mapsto \frac{1}{\tau}$




(c) Choosing a new lattice vector $\tau' = -\frac{1}{\tau}$ in the upper half-plane

**Figure 2.3.:** The modular transformation $\mathcal{S} : \tau \mapsto -\frac{1}{\tau}$

be *rational*. Then

$$\chi_h(-\frac{1}{\tau}) = \sum_{h'} S_h^{h'} \chi_{h'}(\tau) \quad \text{and} \tag{2.1.11}$$

$$\chi_h(\tau+1) = \sum_{h'} T_h^{h'} \chi_{h'}(\tau) \tag{2.1.12}$$

for $h'$ belonging to some finite set of weights. A given torus with its modular parameter $\tau$ is invariant under $\mathcal{S}$ and $\mathcal{T}$ and hence under the whole modular group, since $\mathcal{S}$ and $\mathcal{T}$ generate the modular group.

## 2.2. Fundamental Domain

Applying $\mathcal{S}$ and $\mathcal{T}$ just amounts to a different choice of *fundamental domain*. A fundamental domain with respect to a set of mappings that make up identifications between different elements of a given set consists of exactly those elements of the set such that it is not possible to use one of the mappings to go from one element of the fundamental domain to another. The fundamental domain for the identifications

$$\tau \sim \tau + 1 \tag{2.2.1}$$

$$\tau \sim -\frac{1}{\tau} \tag{2.2.2}$$

is the set of all inequivalent tori. For the first identification, it would just be the infinite strip

$$\mathcal{S}_0 := \left\{ \tau \in \mathbb{C} \mid -\frac{1}{2} < \Re(\tau) \leq \frac{1}{2} , \quad \Im(\tau) > 0 \right\} . \tag{2.2.3}$$

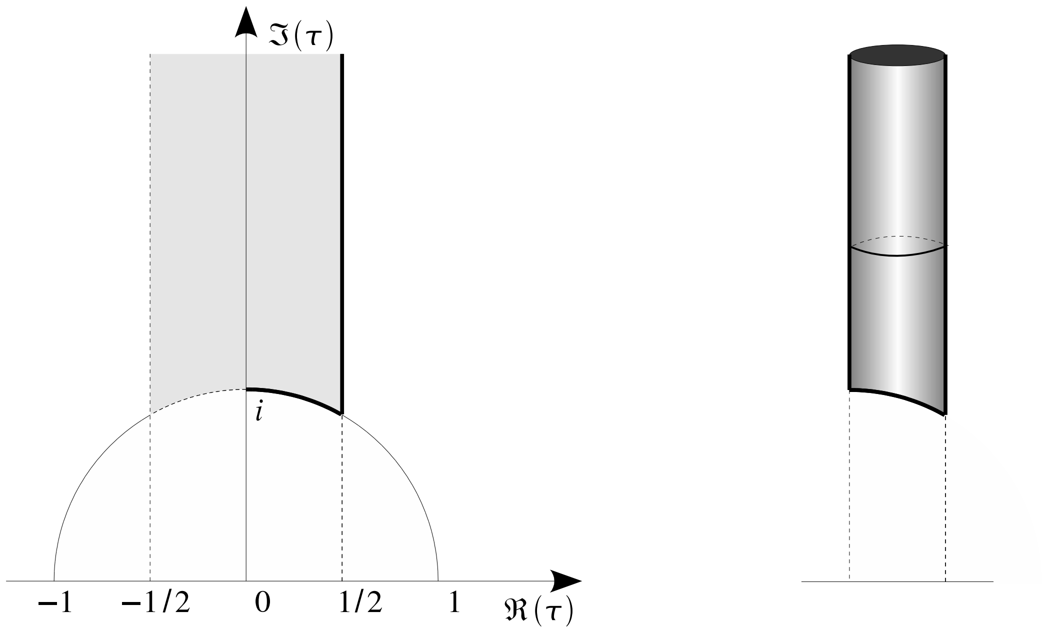
Note that the left boundary is identified with the right one and hence not included in this set.

The second identification identifies points within the unit circle with points outside the unit circle. We choose the points outside the unit circle since it turns out that we still would have a multiple counting if we chose the points inside the unit circle because, among other problems, the corresponding unit circles to neighboring strips overlap. After all, it turns out that a suitable fundamental domain[3] for the modular group $\Gamma = PSL(2,\mathbb{Z})$ is

$$\mathcal{F}_0 := \left\{ \tau \in \mathbb{C} \mid -\frac{1}{2} < \Re(\tau) \leq \frac{1}{2} , \quad \Im(\tau) > 0 , \quad \begin{cases} |\tau| \geq 1 & \text{if } \Re(\tau) \geq 0 \\ |\tau| > 1 & \text{if } \Re(\tau) < 0 \end{cases} \right\} . \tag{2.2.4}$$

It is displayed in figure 2.4. Knowledge of the fundamental domain is essential for *string perturbation theory*: In general, string perturbation theory is a summation over all two-dimensional surfaces, which splits as a sum over all different topologies (just like the loops in Feynman graphs in quantum field theory) and integrations over all different *moduli* of surfaces with a given topology. The topology of genus $g = 1$ corresponds to the torus, and there is one complex modulus, namely the modular parameter $\tau$ discussed above. Therefore, one has to make sure only to be integrating once over all possible tori.

[3] There is a fundamental domain drawer on the internet at http://www.math.lsu.edu/~verrill/fundomain/.




(a) The fundamental domain of the modular group $\Gamma$.

(b) Because of identifications on opposite edges, the fundamental domain may be thought of as rolled up.

**Figure 2.4.:** The fundamental domain

## 2.3. Modular Invariant Partition Function

It has proved interesting to study the dependence of various quantities in the conformal field theory on $\tau$. On a torus, there is the advantage that the operator content of the theory will be constraint from the requirement that the partition function is independent of the choice of $\lambda_1$ and $\lambda_2$, i.e. is invariant under modular transformations. Not every left-right combination of highest weight modules will be physical. The partition function is formally defined as

$$Z = \mathrm{Tr} e^{-\beta H} \tag{2.3.1}$$

with $H$ being the Hamilton operator, which is given by $L_0 + \bar{L}_0$ in a conformal field theory. In the general case, the torus is being twisted before glued together since in general $\Re(\tau) \neq 0$, so one has to take that into account. Space and time directions are defined to run along the real and imaginary axes, respectively. The operator for translations of the system along $\lambda_2$ over a distance $d$ is

$$\mathrm{e}^{-\frac{d}{|\lambda_2|}(H\Im(\lambda_2) - \mathrm{i}P\Re(\lambda_2))} \ . \tag{2.3.2}$$

On a cylinder of circumference $L$,

$$H = \frac{2\pi}{L}(L_0 + \bar{L}_0 - \frac{c}{12}) \quad \text{and} \quad P = \frac{2\pi}{L}(L_0 - \bar{L}_0) \ , \tag{2.3.3}$$

where the constant $\frac{c}{12}$ has its origin in the transformation properties from the plane to the cylinder (cf. section 1.1.6). $H$ and $P$ generate translations along the time and space axis, respectively. Since $\lambda_1 = L$ and using a rescaling of the entire lattice by a factor of $2\pi$ to match conventions, we have

$$Z = \mathrm{Tr} e^{\pi \mathrm{i}\left((\tau-\bar{\tau})(L_0+\bar{L}_0-\frac{c}{12})+(\tau+\bar{\tau})(L_0-\bar{L}_0)\right)} \tag{2.3.4}$$

$$= \mathrm{Tr} e^{2\pi \mathrm{i}\left(\tau(L_0-\frac{c}{24})-\bar{\tau}(\bar{L}_0-\frac{c}{24})\right)} \ . \tag{2.3.5}$$

By setting $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$ and $\bar{q} = \mathrm{e}^{2\pi\mathrm{i}\bar{\tau}}$, we can write the partition function as

$$Z(\tau) = \mathrm{Tr}\left(q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}\right) = (q\bar{q})^{-\frac{c}{24}}\mathrm{Tr}\left(q^{L_0}\bar{q}^{\bar{L}_0}\right) \ . \tag{2.3.6}$$

Conformal invariance implies that the Hilbert space of the conformal field theory splits as a sum of representations of the conformal algebra. Accordingly, the torus partition function is expressible as

$$Z(q) = \sum_{h,\bar{h}} \chi_h(q)\mathcal{N}_{h\bar{h}}\chi_{\bar{h}}(\bar{q}) \ , \tag{2.3.7}$$

where $\chi_h(q)$ is the (Virasoro) character of the representation with highest weight $h$, defined as

$$\chi_h(q) = \mathrm{Tr}_{V(h,c)} q^{L_0-\frac{c}{24}} \ , \tag{2.3.8}$$

where $\chi_h(q) \in q^h\mathbb{Z}[[q]]$ or, for logarithmic theories, $\chi_h(q) \in q^h\mathbb{Z}[[q]][\log(q)]$, if one chooses to allow logarithmic terms in the character functions, and where the trace is over all (positive norm) states in the representation corresponding to weight $h$. The coefficients $\mathcal{N}_{h\bar{h}}$ are all integers and $\mathcal{N}_{00} = 1$, since the vacuum is assumed to be unique. A symmetry algebra is *maximally extended* if $\mathcal{N}_{h\bar{h}}$ is diagonal. In fact, Werner Nahm [Nah91] proved the statement by John Cardy [Car86], which holds for diagonalizable $L_0$, that conformal invariance of a theory on $S^2$ implies modular invariance of the theory's partition function on the torus. But it is assumed that this also holds for logarithmic theories with non-trivial Jordan cells, i.e. where $L_0$ is not diagonalizable. In any case, the so-called *diagonal invariant* $\mathcal{N}_{ij} = \delta_{ij}$, where $\mathcal{N}_{ij} := \mathcal{N}_{h_i\bar{h}_j}$, is always a solution to the problem of determining a modular invariant partition function. For example, a modular invariant partition function for the conformal field theory at central charge $c = \frac{1}{2}$, which is a minimal model and has three Virasoro representations – one each at $h = 0$, $h = \frac{1}{2}$ and $h = \frac{1}{16}$ – is obtained by choosing the three ground states $|h = 0, \bar{h} = 0\rangle$, $|\frac{1}{2}, \frac{1}{2}\rangle$ and $|\frac{1}{16}, \frac{1}{16}\rangle$. The three primary fields $\mathbb{1}$, $\epsilon$ and $\sigma$ create these ground states from the vacuum, respectively. This theory is the Ising model (cf. section 1.3.3). The possible modular invariant torus partition functions for the minimal Virasoro models of central charges $c_{p,p'} = 1 - 6\frac{(p-p')^2}{pp'}$, $p, p' \in \mathbb{Z}_{\geq 2}$ and coprime, have been classified in [CIZ87b]. This has been termed *ADE classification* since all solutions are labelled by simply-laced Lie algebras. With this result, all unitary models of CFT with $c < 1$ are explicitly known. The case $c = 1$, corresponding to the compactified free boson theories, also shows a connection to the $ADE$ classification [Gin88]. The classification of the $c > 1$ theories as well as the non-minimal $c < 1$ theories, is still an open question. Most remarkably, $ADE$ occurs also in many other classification tasks, e.g. classification of modular invariants of conformal field theories with Lie algebra symmetry, classification of the finite $SU(2)$ subgroups. The $ADE$ algebras also directly turn up in the fermionic character expressions of rational conformal field theories, as we will see later in this thesis. The origin of all this still remains a mystery.

## 2.4. Fusion Rules and the Verlinde Formula

In an operator product expansion, the constants $C_{ijk}$, which are the same as in the three-point function, satisfy certain selection rules imposed by the Virasoro algebra. These are called *fusion rules* [BPZ84] and are written as

$$[i] \times [j] = \sum_{k} N_{ijk}[k] \ , \tag{2.4.1}$$

where $[i]$ denotes the conformal family corresponding to the primary field $\Phi_i(z)$ and $N_{ijk} \in \mathbb{Z}_{\geq 0}$. The fusion rules thus determine which conformal families can occur in an operator product expansion. In the case of the minimal models which consist of only a finite number of primary fields, the sum on the right hand side of (2.4.1) is truncated to include only a finite number of terms.

Erik Verlinde discovered a remarkable connection [Ver88] later proven by Moore and Seiberg [MS88] between the coefficients $N_{ijk}$ and the representation of the modular transformation $\mathcal{S}$ on the space of character functions, the so-called *Verlinde formula*

$$N_{ijk} = \sum_{l} \frac{S_{il}S_{jl}S_{kl}}{S_{0l}} \ . \tag{2.4.2}$$

It implies that the fusion rules are completely determined by the behavior of the characters under transformations of the modular group and, conversely, that the modular properties of the characters can be derived from the fusion rules. The matrix $S$ simultaneously diagonalizes the fusion rules $(N_i)_{jk}$ for all fields $\Phi_i$. For logarithmic conformal field theories, things are not that simple. Recently, progress has been made in this direction [Knu06, Knu07].

## 2.5. $\Theta$- and $\eta$-functions and their Modular Transformation Properties

The *Jacobi-Riemann $\Theta$-functions* and the *affine $\Theta$-functions* are defined by

$$\Theta_{\lambda,k}(\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{(2kn+\lambda)^2}{4k}} \tag{2.5.1}$$

and

$$(\partial\Theta)_{\lambda,k}(\tau) = \sum_{n\in\mathbb{Z}} (2kn+\lambda) q^{\frac{(2kn+\lambda)^2}{4k}} \tag{2.5.2}$$

with $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$. $\lambda \in \frac{\mathbb{Z}}{2}$ is called the *index* and $k \in \frac{\mathbb{Z}_{\geq 1}}{2}$ the *modulus*. The $\Theta$-functions satisfy the symmetries

$$\Theta_{\lambda,k} = \Theta_{-\lambda,k} = \Theta_{\lambda+2k,k} \quad \text{and} \tag{2.5.3}$$

$$(\partial\Theta)_{-\lambda,k} = -(\partial\Theta)_{\lambda,k} \ . \tag{2.5.4}$$

The *Dedekind $\eta$-function* is defined as

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \ . \tag{2.5.5}$$

The Jacobi-Riemann $\Theta$-functions and the Dedekind $\eta$-function are modular forms of weight 1/2, while the affine $\Theta$-functions have modular weight $\frac{3}{2}$. A *modular form of weight $k$* is defined by the relation

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \epsilon(a, b, c, d)(c\tau + d)^k f(\tau) \tag{2.5.6}$$

for $\tau \in \mathbb{C}$ and $|\epsilon(a, b, c, d)| = 1$ and with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $f$ being a holomorphic function on the upper half-plane which is also holomorphic at the *cusp*, i.e. is holomorphic as $\tau \to \mathrm{i}\infty$. The modular transformation properties of the $\Theta$- and $\eta$-functions for those cases of $\lambda$ and $k$ that are needed in this thesis are

$$\Theta_{\lambda,k}(-\frac{1}{\tau}) = \sqrt{\frac{-\mathrm{i}\tau}{2k}} \sum_{\lambda'=0}^{2k-1} \mathrm{e}^{\mathrm{i}\pi \frac{\lambda\lambda'}{k}} \Theta_{\lambda',k}(\tau) \quad \text{for } \lambda \in \mathbb{Z} \tag{2.5.7}$$

$$\Theta_{\lambda,k}(\tau + 1) = \mathrm{e}^{\mathrm{i}\pi \frac{\lambda^2}{2k}} \Theta_{\lambda,k}(\tau) \quad \text{for } \lambda - k \in \mathbb{Z} \tag{2.5.8}$$

$$\eta(-\frac{1}{\tau}) = \sqrt{-\mathrm{i}\tau}\,\eta(\tau) \tag{2.5.9}$$

$$\eta(\tau + 1) = \mathrm{e}^{\frac{\pi \mathrm{i}}{12}} \eta(\tau) \ . \tag{2.5.10}$$

The functions $\chi_{\lambda,k} = \frac{\Theta_{\lambda,k}}{\eta}$, which often turn up as summands in character functions in the chapter 3.1, are thus modular forms of weight zero with respect to the main-congruence subgroup $\Gamma(N)$ of the modular group $PSL(2\mathbb{Z})$. Meny details about $\Theta$-functions may be found in [Igu72, Akh90].

# 3. Characters

## 3.1. General Virasoro Characters

The Hilbert space of the theory decomposes as

$$\mathcal{H} = \bigoplus_{h,\bar{h}} V(h,c) \otimes V(\bar{h},c) \ . \tag{3.1.1}$$

For simplicity, we consider only the holomorphic sector of the theory. If the antiholomorphic sector is just a copy of the holomorphic one, the arguments given are the same for the antiholomorphic sector. But it is also possible to have a theory where this is not the case. For example, in the case of heterotic strings, both sectors are very different from each other.

The *character* of an indecomposable $\mathcal{V}$-module $V$ is defined by

$$\chi_V(\tau) := \mathrm{Tr} e^{2\pi i\tau(L_0 - \frac{c}{24})} \ . \tag{3.1.2}$$

with $c$ being the central charge and $L_0$ the Virasoro zero mode. The characters of the representations are an essential ingredient for a conformal field theory. Since $L_0$ corresponds to the Hamiltonian of the (chiral half of the) system, the energy spectrum (at least certain sectors) is encoded in the character. The trace is usually taken over an irreducible highest weight representation and the factor $q^{-\frac{c}{24}}$ guarantees the needed linear behavior under modular transformations. By setting $q := e^{2\pi i\tau}$, (3.1.2) leads to

$$\chi_V(q) = q^{-\frac{c}{24}} \sum_h q^h \dim \text{ eigenspace}(L_0^d, h) \ , \tag{3.1.3}$$

where $L_0^d$ is the diagonalizable summand of the possibly non-diagonalizable $L_0$. To compute the character of a Verma module, we have to compute the number of linear independent states at a given level $k$. We can grade the Verma module by its $L_0$ eigenvalue:

$$V(h,c) = \bigoplus_N V_N(h,c) \tag{3.1.4}$$

with

$$V_n(h,c) = \left\langle \left\{ L_{-n_1} L_{-n_2} \cdots L_{-n_k} |h\rangle \mid k \in \mathbb{Z}_{\geq 0},\ n_{i+1} \geq n_i,\ \sum\nolimits_{i=1}^{k} n_i = N \right\} \right\rangle \ . \tag{3.1.5}$$

Thus, the number of distinct, linear independent states at leven $N$ is given by the number $p(N)$ of additive partitions of the integer $N$. The generating function for the number of partitions is

$$\frac{1}{\phi(q)} \equiv \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{n=0}^{\infty} p(N) q^n \ , \tag{3.1.6}$$

where $\phi(q)$ is the Euler function. Hence the character is given by

$$\chi_{V(h,c)} = q^{h-\frac{c}{24}} \sum_{k=0}^{\infty} p(k) q^k \ . \tag{3.1.7}$$

Dedekind's $\eta$ function $\eta(q) = q^{1/24}\phi(q)$ is conventionally used because it simplifies the analysis of a character function under modular transformations:

$$\chi_{V(h,c)} = \frac{q^{\frac{1-c}{24}}}{\eta(q)} q^h \ . \tag{3.1.8}$$

This series is convergent if $|q| < 1$, i.e. $\tau \in \mathfrak{H}$ (upper half-plane). If $V(h,c)$ already is an irreducible representation, i.e. is non-degenerate, then this is its character. If not, the characters $\chi_{r,s}$ of irreducible representations $M(h_{r,s},c)$ can be read off the above embedding structure.

## 3.2. Bosonic and Fermionic Expressions for Characters

There is more than one way to write a character in a closed form. Among these are the so-called bosonic and fermionic character representations. $\frac{1}{(q)_\infty}$ is just the character of a free, chiral boson with momenta $p \in \mathbb{Z}_{\geq 1}$. If the representation space is truncated by a subsequent subtraction and addition of singular vectors, it well be encoded in the numerator. Thus, these character expressions have been termed bosonic. Aside from the bosonic expressions, there are the fermionic quasi-particle sum representations for a character, also called fermionic expressions, which first appeared under this name in [KM93]. These are interesting from both a mathematical and a physical point of view and first occured in an especially simple form in the context of the Rogers-Schur-Ramanujan identities [Rog94, Sch17, RR19] (for $a \in \{0,1\}$)

$$\sum_{n=0}^{\infty} \frac{q^{n(n+a)}}{(q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1-a})(1-q^{5n-4+a})} \tag{3.2.1}$$

with the so-called *$q$-analogues*

$$(x;q)_n := \prod_{i=0}^{n-1}(1-q^i x) \quad \text{and} \quad (q)_n := (q;q)_n = \prod_{i=1}^{n}(1-q^n) \tag{3.2.2}$$

of the Pochhammer symbol and the classical factorial function, respectively, and by definition

$$(q)_0 := 1 \quad \text{and} \quad (q)_\infty := \lim_{n\to\infty} (q)_n \ , \tag{3.2.3}$$

the latter being the $q$-analogues of the classical gamma function. Note that $(q)_\infty$ is up to factor of $q^{\frac{1}{24}}$ the modular form $\eta(\tau)$ with $q = \mathrm{e}^{2\pi \mathrm{i}\tau}$, the Dedekind $\eta$-function. These identities coincide with the two characters of the minimal model $\mathcal{M}(2,5)$ with central charge $c = -\frac{22}{5}$, which represents the Yang-Lee model (up to an overall factor of $q^\alpha$ for some $\alpha \in \mathbb{C}$). It is the smallest minimal model and contains only two primary operators: the identity $\mathbb{1}$ of dimension $(h,\bar{h})=(0,0)$ and another operator $\varPhi$ of dimension $(-\frac{1}{5},-\frac{1}{5})$. By using Jacobi's triple product identity [Jac29], defined for $z\neq 0$ and $|q|<1$ (see appendix B or [And84]) as

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = \prod_{n=1}^{\infty}(1-q^{2n})(1+zq^{2n-1})(1+z^{-1}q^{2n-1}) \ , \tag{3.2.4}$$

the r.h.s. of (3.2.1) can be transformed to give a simple example of what is called a *bosonic-fermionic q-series identity*:

$$\sum_{n=0}^{\infty}\frac{q^{n(n+a)}}{(q)_n} = \frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty}\left(q^{n(10n+1+2a)}-q^{(5n+2-a)(2n+1)}\right) \tag{3.2.5}$$

An instructive proof of the Jacobi triple product identity is given in appendix B by comparison of the characters computed from a fermionic basis of the irreducible vacuum representation of charged free fermion system with the character computed from a bosonic basis of the same representation obtained by bosonization.

In general, it is always possible to write a minimal model character in a product form and thus to obtain a Rogers-Ramanujan-type identity if $p=2s$ or $p'=2r$, as has been demonstrated by Philippe Christe in [Chr91]. To see this, one employs the Jacobi triple product identity (3.2.4) with the replacements $q\mapsto q^{\frac{pp'}{2}}$ and $z\mapsto -q^{rs-\frac{pp'}{4}}$. Product forms are also possible in the case $p=3s$ or $p'=3r$, but to show this, the so-called *Watson identity* [Wat29] (see also [GR90, ex. 5.6]) has to be used instead of the Jacobi identity. Christe also proved in the same article that for other minimal model characters, no product forms of this type exist.

The *bosonic expressions* on the r.h.s. of (3.2.5) correspond to two special cases of the general character formula (3.4.10) for minimal models by Rocha-Caridi [RC84]. Explictly, they are given by

$$\begin{aligned}\chi_{1,1}^{5,2} =&1+q^2+q^3+q^4+q^5+2q^6+2q^7+3q^8+3q^9+4q^{10}\\ &+4q^{11}+6q^{12}+6q^{13}+8q^{14}+9q^{15}+11q^{16}+12q^{17}+15q^{18}+16q^{19}+20q^{20}+\dots\end{aligned}$$

and

$$\begin{aligned}\chi_{1,2}^{5,2} =&1+q+q^2+q^3+2q^4+2q^5+3q^6+3q^7+4q^8+5q^9+6q^{10}\\ &+7q^{11}+9q^{12}+10q^{13}+12q^{14}+14q^{15}+17q^{16}+19q^{17}+23q^{18}+26q^{19}+31q^{20}+\dots\ .\end{aligned}$$

Note that the coefficient of $q$ is zero because the vacuum is invariant under $L_n$, $n\in\{-1,0,1\}$. Since the right hand side of (3.2.5) is computed by eliminating null states from the Hilbert space of a free chiral boson [FF83], it is referred to as bosonic form. Its signature is the alternating sign, which reflects the subtraction of null vectors. The factor $(q)_\infty$ keeps track of the free action of the Virasoro 'raising' modes. Furthermore, it can be expressed in terms of $\varTheta$-functions (cf. 2.5), which directly point out the modular transformation properties of the character.

On the other hand, the left-hand side of (3.2.1) has a direct fermionic quasi-particle interpretation for the states and hence is called fermionic sum representation. In the first systematic study of fermionic expressions [KKMM93b], sum representations for all characters of the unitary Virasoro minimal models and certain non-unitary minimal models were given. The list of expressions was augmented to all $p$ and $p'$ and certain $r$ and $s$ in [BMS98]. Eventually, the fermionic expressions for the characters of all minimal models were summarized in [Wel05]. Such a fermionic expression, which is a generalization of the left hand side of (3.2.1), is a linear combination of fundamental fermionic forms. A *fundamental fermionic form* [BMS98, Wel05, DKMM94] is

$$\sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^r \\ \text{restrictions}}} \frac{q^{\vec{m}^t A\vec{m}+\vec{b}^t\vec{m}+c}}{\prod_{i=1}^{j}(q)_i} \prod_{i=j+1}^{r} \begin{bmatrix} g(\vec{m}) \\ m_i \end{bmatrix}_q \tag{3.2.6}$$

with $A\in M_r(\mathbb{Q})$, $\vec{b}\in\mathbb{Q}^r$, $c\in\mathbb{Q}$, $0\leq j\leq r$, $g$ a certain linear, algebraic function in the $m_i, 1\leq i\leq r$, and the $q$-binomial coefficient defined as

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{cases} \frac{(q)_n}{(q)_m(q)_{n-m}} & \text{if } 0\leq m\leq n \\ 0 & \text{otherwise} \end{cases} . \tag{3.2.7}$$

The sum over $\vec{m}$ is an abbreviation and implies that each component $m_i$ of $\vec{m}$ is to be summed over independently.[1]

Fermionic character expressions in conformal field theory have various origins. Aside from the Rogers-Ramanujan identities, they first appeared in the representation theory of Lie algebras as the Lepowsky-Primc formulae [LW81b, LW81a, LW84, LW85, LP85, FZ85, FNO92] and Andrews-Gordon identities [And74, NRT93]. They also arise from thermodynamic Bethe ansatz analysis of integrable perturbations of conformal field theory [KM90, KM92] resulting in dilogarithm identities (cf. chapter 4) which may be lifted back [Ter92] to fermionic expressions, from the scaling limit of spin chains and $ADE$ generalizations of Lepowsky-Primc [KM93, DKMM94] and from spinon bases for WZW models [BPS94, BLS95a, BLS95b]. All these different origins will be discussed in detail in the following sections.

## 3.3. Nahm's Conjecture

The question of how *$q$-hypergeometric series* (i.e. series of the form $\sum_{n=0}^{\infty} A_n(q)$ where $A_0(q)$ is a rational function and $A_n(q) = R(q,q^n)A_{n-1}(q)\forall n\geq 1$ for some rational function $R(x,y)$ with $\lim_{x\to 0}\lim_{y\to 0} R(x,y) = 0$) are related to modular forms or modular functions is an almost completely unsolved problem. But there is a conjecture by Werner Nahm (see e.g. [Nah04]), which involves dilogarithms and torsion elements of the so-called *Bloch group* as well as rational conformal field theories. If $j=r$ in (3.2.6), then the fundamental fermionic form reduces (with a rescaling $A\mapsto \frac{1}{2}A$)[2] to the $q$-hypergeometric series

$$f_{A,\vec{b},c}(\tau) = \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^r \\ \text{restrictions}}} \frac{q^{\frac{1}{2}\vec{m}^t A\vec{m}+\vec{b}^t\vec{m}+c}}{(q)_{\vec{m}}} . \tag{3.3.1}$$

[1] The constant $c$ is not to be confused with the central charge $c_{p,p'}$.

[2] This rescaling has only been done in this section, since it makes the discussion of matrices $A$ and their inverses easier. For the rest of this thesis, this rescaling is not necessary.

*Nahm's conjecture* has no complete answer to this, but it makes a prediction which matrices $A \in M_r(\mathbb{Q})$ can occur such that (3.3.1) is a modular function, i.e. whether there exist suitable $\vec{b} \in \mathbb{Q}^r$ and $c \in \mathbb{Q}$ for a given matrix $A$. In particular, such a function can only be modular when all solutions to a certain system of algebraic equations depending on the coefficients of $A$, namely

$$1 - x_i = \prod_{j=1}^{r} x_j^{A_{ij}} \Longleftrightarrow \sum_{j=1}^{r} A_{ij} \log(x_j) = \log(1 - x_i) \tag{3.3.2}$$

(the same we will also encounter in chapter 4), yield elements $\sum_i [x_i]$ of finite order in the Bloch group of the algebraic numbers [Nah04].

The physical significance of this is that one expects that all the $q$-hypergeometric series which are modular functions are characters of rational conformal field theories. Given a matrix $A$, the modular forms for the predicted possible combinations of vectors and constants span a finite-dimensional vector space that is invariant under $SL(2,\mathbb{Z})$ for bosonic CFTs (or under $\Gamma_0(2) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}) \mid c \in 2\mathbb{Z}\}$ for fermionic CFTs), i.e. the set of characters generated in this way forms a finite-dimensional representation of the modular group, which is just the definition of rationality of a conformal field theory. Indeed, this is just what we will find in the subsequent analysis in this thesis: The admissible matrices of rank one and two correspond to rational theories, most of them to the minimal models.

In general, there exist fermionic expressions for all characters of the minimal models. However, all but a finite number are not known to be of the type (3.3.1). Instead, they consist of finite linear combinations of fundamental fermionic forms (3.2.6), i.e. they involve finite $q$-binomial coefficients. But nevertheless, it is usually possible to express all characters of a given minimal model in terms of the same matrix $A$, albeit the choice of the matrix for that given model is in general not unique. We will comment more on that in the subsequent sections.

Note furthermore that the series of so-called triplet $\mathcal{W}$-algebras, which are logarithmic conformal field theories to be discussed later in this thesis, was shown to be rational (in a broader sense to be defined later) with respect to its extended $\mathcal{W}$-symmetry algebra. These theories are not rational with respect to the Virasoro algebra alone as the symmetry algebra. By presenting fermionic sum-representations of Nahm type (3.3.1) for the characters of the whole series of $\mathcal{W}(2, 2p-1, 2p-1, 2p-1)$ triplet algebras ($p \geq 2$), thus leading to a new infinite set of bosonic-fermionic $q$-series identities, we further support Nahm's conjecture and provide further evidence that the triplet algebra series are well-defined new animals in the zoo [Flo03] of rational conformal field theories.

There are also fermionic expressions for characters of other theories than the above mentioned, including for example the Kač-Peterson characters of the affine Lie algebra $A_1^{(1)}$ [KP84], which we will also discuss later and for which we also present new fermionic expressions.

A lot of matrices $A$, among them in particular one infinite series, have been found for which Nahm's conjecture suggests that they should lead to modular forms. However, a complete search has only been achieved for rank one and two matrices. Some of them are related to the *Dynkin diagrams* of the type $A$, $D$, $E$ or $T$, corresponding to the simple Lie algebras.[3]

[3]Watch the notation problem: The matrix in the exponent of the fermionic character expression is conventionally labeled $A$. This is not to be confused with the $A$ series of Dynkin diagrams.

These diagrams, as shown in fig. 3.1, have $r$ vertices if they are called $X_r$, where $X$ is to be replaced by $A$, $D$, $E$ or $T$. In many cases, the matrix $A$ in the quadratic form in (3.3.1) is just twice the inverse Cartan matrix of a Dynkin diagram. On the other hand, it may also be half the Cartan matrix itself. These two cases are to be regarded as the special cases of another class, namely $A = C_{X_r} \otimes C_{Y_s}^{-1}$, where $X_r, Y_s \in \{A, D, E, T\}$. Note that the *Cartan matrix* is in one-to-one correspondence with a Dynkin diagram: For each vertex $i$ that is connected to a vertex $j$ $(i, j \in \{1, \ldots, r\})$, set $A_{ij}A_{ji}$ equal to the number of lines connecting these two vertices with the restriction that $A_{ij}, A_{ji} \in \mathbb{Z}_{\leq 1}$, set $A_{ii} = 0\ \forall i \in \{1, \ldots, r\}$. All the other entries are zero. An exception to this is $C_{T_r}$, which is equal to $C_{A_r}$ in all components but in the lower right one: $(C_{T_r})_{rr} = 1$. $T_r$ is the so-called *tadpole graph* corresponding to $A_{2r}$ folded in the middle such that vertices are pairwise identified. Many $ADE$ related matrices of rank greater than two have also been found to correspond to rational conformal field theories, especially the inverse Cartan matrices. Examples of this kind will be discussed in section 3.5. Modular forms with matrices of the second class can be found e.g. in Kač-Peterson characters later in this thesis, while the first class is common to minimal models. But there are also other types, some of which don't seem to fit in this pattern.

The following is a list of the matrices of rank one and two which lead to modular forms when inserted into (3.3.1):

**Rank** $r = 1$**:** Two following combinations yield modular forms:

$A = 2$**:** $\vec{b} = 0,\ c = -\frac{1}{60}$ or $\vec{b} = 1,\ c = \frac{11}{60}$.

$A = 1$**:** $\vec{b} = 0,\ c = -\frac{1}{48}$ or $\vec{b} = \frac{1}{2},\ c = \frac{1}{24}$ or $\vec{b} = -\frac{1}{2},\ c = \frac{1}{24}$.

$A = \frac{1}{2}$**:** $\vec{b} = 0,\ c = -\frac{1}{40}$ or $\vec{b} = \frac{1}{2},\ c = \frac{1}{40}$.

**Rank** $r = 2$**:** The following matrices $A$ fulfill the condition of Nahm's conjecture[4] and indeed, for each of those, there are several $\vec{b}$ and $c$ that constitute modular forms:

$$\begin{pmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{pmatrix},\ \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},\ \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix},\ \begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix},\ \begin{pmatrix} 2 & 1 \\ 1 & \frac{3}{2} \end{pmatrix},\ \begin{pmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{4}{3} \end{pmatrix} \tag{3.3.3}$$

corresponding respectively to effective central charges $1,\ \frac{3}{4},\ \frac{7}{10},\ \frac{4}{7},\ \frac{5}{7},\ \frac{4}{5}$ as well as the inverses of these matrices, having $c_{\text{eff}}(A^{-1}) = r - c_{\text{eff}}(A)$. The central charges can be computed via the sum of dilogarithm functions evaluated at the solutions of (3.3.2), see chapter 4.

For level three, the classification is still an unsolved task. However, Don Zagier [Zag06] searched positive definite $3 \times 3$ matrices with integer coefficients which smaller than or equal to ten and found about forty matrices which satisfy the conditions of Nahm's conjecture.

# 3.4. Characters of Minimal Models

## 3.4.1. Bosonic Character Expressions for Minimal Models

To compute the characters for a given minimal model, one needs to know the embedding structure of the irreducible representations, which is given in section 1.2.5. The highest

[4]Don Zagier states [Zag06] that he tested matrices $A = \frac{1}{m}\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ with integers $a, b, c, m \leq 100$ and found no other matrices than the ones listed here.

weights of the model (which correspond to the conformal dimensions of the primary fields) can be read off the Kač table. As can be seen by the Kač determinant formula and the embedding structure in the case of minimal models, the irreducible representations are given by

$$M_{r,s} = V_{r,s}/(V_{r,-s} + V_{r,2p-s}) \tag{3.4.1}$$

To compute the Virasoro character, one has to further investigate the sum of the two submodules. In each of the two submodules, there are another two submodules. By the symmetry property

$$h_{r,s} = h_{p'-r,p-s} \tag{3.4.2}$$

it turns out that the pair of submodules in $V_{r,-s}$ (see (1.2.24)) is identical to the pair of submodules in $V_{r,2p-s}$. This means

$$(V_{r,-s} + V_{r,2p-s}) = (V_{r,s} \cup V_{r,2p-s})/(V_{r,s+2p} + V_{r,s-2p}) \tag{3.4.3}$$

When iterating this procedure, it turns out that $M_{r,s}$ is given by an infinite subtraction and addition of submodules

$$M_{r,s} = V_{r,s} - (V_{r,-s} \cup V_{r,2p-s}) + (V_{r,s+2p} \cup V_{r,s-2p}) - \dots \ . \tag{3.4.4}$$

Since the Virasoro character is

$$\chi_{V(h,c)} = \frac{q^{\frac{1-c}{24}}}{\eta(q)} q^h \ , \tag{3.4.5}$$

as shown above, one has to subtract and add an infinite number of terms. If there was just one submodule in the original Verma module which would contain a singular vector whose Verma module would in turn contain a singular vector and so on, then one would just have to subtract a single term, as in the case of the logarithmic models. In the case of the minimal models, it is necessary to subtract two terms, because there are two singular vectors. But since the pairs of submodules of each submodule are identical, these are doubly subtracted. Hence, one then has to correct this and add the two terms corresponding to one of those pairs again, and so on. It is possible to set up a formula for the character of the highest weight representations of a minimal model as an infinite series of additions and subtractions of Virasoro characters, the *Rocha-Caridi character* [RC84]

$$\chi^{p,p'}_{r,s}(q) = \frac{q^{\frac{1-c}{24}}}{\eta(q)} \left(q^{h_{r,s}} - (q^{h_{r,-s}} + q^{h_{r,2p-s}}) + (q^{h_{r,s+2p}} + q^{h_{r,s-2p}}) - \dots\right) \tag{3.4.6}$$

$$= \frac{q^{\frac{1-c}{24}}}{\eta(q)} \sum_{n\in\mathbb{Z}} \left(q^{h_{r,s+2np}} - q^{h_{r,2np-s}}\right) \tag{3.4.7}$$

$$= \frac{1}{\eta(q)} \sum_{n\in\mathbb{Z}} \left(q^{\frac{(pr-p's-2npp')^2}{4pp'}} - q^{\frac{(pr+p's-2npp')^2}{4pp'}}\right) \tag{3.4.8}$$

$$= \frac{1}{\eta(q)} \sum_{n\in\mathbb{Z}} \left(q^{\frac{(2kn+\lambda)^2}{4k}} - q^{\frac{(2kn+\lambda')^2}{4k}}\right) \tag{3.4.9}$$

with $\lambda = pr - p's$, $\lambda' = pr + p's$ and $k = pp'$. In terms of theta functions (see 2.5),

$$\chi_{\lambda,k} = \frac{\Theta_{\lambda,k} - \Theta_{\lambda',k}}{\eta} \ . \tag{3.4.10}$$

The Rocha-Caridi characters have the symmetries

$$\chi^{p,p'}_{r,s} = \chi^{p',p}_{s,r} = \chi^{p,p'}_{p'-r,p-s} \;, \tag{3.4.11}$$

corresponding to symmetries of the Kač table, and

$$\chi^{p,\alpha p'}_{\alpha r,s} = \chi^{\alpha p,p'}_{r,\alpha s} \quad \alpha \in \mathbb{Z}_{\geq 1} \;,\; \langle p, \alpha p' \rangle = \langle \alpha p, p' \rangle = 1 \;. \tag{3.4.12}$$

### 3.4.2. Fusion in Minimal Models

Since the space of characters of the irreducible modules forms a natural representation of the modular group, the minimal models are rational in the sense that the representation of the modular group $PSL(2,\mathbb{Z})$ on the characters is finite-dimensional. The property that the space of characters forms a finite-dimensional representation of the modular group is regarded as a definition of *rational conformal field theories*. In particular, the partition function

$$Z(\tau) := \sum_{i,j} \mathcal{N}_{i,j} \chi^*_{h_i}(\tau) \chi_{h_j}(\tau) \tag{3.4.13}$$

of the complete theory is modular invariant if the integers are chosen appropriately. The 'diagonal' choice $N_{i,j} \sim \delta_{i,j}$ is always a solution. For a rational conformal field theory, the operator product expansion of fields in the complete theory closes in finitely many families. Namely, the fusion rules (cf. section 2.4) reduce to [BPZ84]

$$[h_{r,s}] \times [h_{t,u}] = \sum_{\substack{m=1+|r-t| \\ m+r+t\equiv 1 \pmod 2 \\ m \in (2\mathbb{Z}+1+|r-t|)}}^{m_{\max}} \; \sum_{\substack{n=1+|s-u| \\ n+s+u\equiv 1 \pmod 2 \\ n \in (2\mathbb{Z}+1+|s-u|)}}^{n_{\max}} [h_{m,n}] \tag{3.4.14}$$

with

$$m_{\max} = \min\{r+t-1, 2p'-1-r-t\} \tag{3.4.15}$$

$$n_{\max} = \min\{s+u-1, 2p-1-s-u\} \;. \tag{3.4.16}$$

### 3.4.3. Fermionic Expressions for Minimal Model Characters

The matrices of the rank one and two case of section 3.3 can be associated with the characters of certain minimal models via the fermionic form (3.3.1). For example, the characters of the *critical Ising model* (cf. section 1.3.3) $\mathcal{M}(4,3)$ have the fermionic expressions

$$\chi_{1,1}^{4,3} = \sum_{\substack{m=0 \\ m\equiv 0 \pmod 2}}^{\infty} \frac{q^{\frac{m^2}{2}-\frac{1}{48}}}{(q)_m} = \frac{\Theta_{1,12}-\Theta_{7,12}}{\eta} \tag{3.4.17}$$

$$\chi_{2,1}^{4,3} = \sum_{\substack{m=0 \\ m\equiv 1 \pmod 2}}^{\infty} \frac{q^{\frac{m^2}{2}-\frac{1}{48}}}{(q)_m} = \frac{\Theta_{5,12}-\Theta_{11,12}}{\eta} \tag{3.4.18}$$

$$\chi_{1,2}^{4,3} = \sum_{m=0}^{\infty} \frac{q^{\frac{m^2}{2}+\frac{m}{2}+\frac{1}{24}}}{(q)_m} = \frac{\Theta_{-2,12}-\Theta_{10,12}}{\eta} \tag{3.4.19}$$

$$= \sum_{\substack{m=0 \\ m\equiv a \pmod 2}}^{\infty} \frac{q^{\frac{m^2}{2}-\frac{m}{2}+\frac{1}{24}}}{(q)_m}\,, \quad a\in\{0,1\} \tag{3.4.20}$$

whereas $\mathcal{M}(5,3)$, which describes the continuum limit of the *critical $O(n)$ model* (a generalization of the Ising model analytically continued to $n=-1$) [Nie84, DFSZ87a, DFSZ87b] has the fermionic character expressions

$$\chi_{1,1}^{5,3} = \sum_{\substack{m=0 \\ m\equiv 0 \pmod 2}}^{\infty} \frac{q^{\frac{m^2}{4}+\frac{m}{2}+\frac{1}{40}}}{(q)_m} = \frac{\Theta_{2,15}-\Theta_{8,15}}{\eta} \tag{3.4.21}$$

$$\chi_{2,1}^{5,3} = \sum_{\substack{m=0 \\ m\equiv 1 \pmod 2}}^{\infty} \frac{q^{\frac{m^2}{4}+\frac{m}{2}+\frac{1}{40}}}{(q)_m} = \frac{\Theta_{7,15}-\Theta_{13,15}}{\eta} \tag{3.4.22}$$

$$\chi_{1,2}^{5,3} = \sum_{\substack{m=0 \\ m\equiv 0 \pmod 2}}^{\infty} \frac{q^{\frac{m^2}{4}-\frac{1}{40}}}{(q)_m} = \frac{\Theta_{-1,15}-\Theta_{11,15}}{\eta} \tag{3.4.23}$$

$$\chi_{1,3}^{5,3} = \sum_{\substack{m=0 \\ m\equiv 1 \pmod 2}}^{\infty} \frac{q^{\frac{m^2}{4}-\frac{1}{40}}}{(q)_m} = \frac{\Theta_{-4,15}-\Theta_{14,15}}{\eta} \tag{3.4.24}$$

and the minimal model $\mathcal{M}(5,2)$ corresponding to the *Yang-Lee model* [Fis78, Car85] has

$$\chi_{1,1}^{5,2} = \sum_{m=0}^{\infty} \frac{q^{m^2+m+\frac{11}{60}}}{(q)_m} = \frac{\Theta_{3,10}-\Theta_{7,10}}{\eta} \tag{3.4.25}$$

$$\chi_{1,2}^{5,2} = \sum_{m=0}^{\infty} \frac{q^{m^2-\frac{1}{60}}}{(q)_m} = \frac{\Theta_{1,10}-\Theta_{9,10}}{\eta}\,. \tag{3.4.26}$$

In the rank two case, the minimal model $\mathcal{M}(7,2)$ has the characters

$$\chi_{1,1}^{7,2} = \sum_{\vec{m}\in(\mathbb{Z}_{\geq 0})^2} \frac{q^{\vec{m}^t\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)\vec{m}+\left(\begin{smallmatrix}2\\1\end{smallmatrix}\right)^t\vec{m}+\frac{17}{42}}}{(q)_{\vec{m}}} \tag{3.4.27}$$

$$\chi_{1,2}^{7,2} = \sum_{\vec{m}\in(\mathbb{Z}_{\geq 0})^2} \frac{q^{\vec{m}^t\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)\vec{m}+\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)^t\vec{m}+\frac{5}{42}}}{(q)_{\vec{m}}} \tag{3.4.28}$$

$$\chi_{1,3}^{7,2} = \sum_{\vec{m}\in(\mathbb{Z}_{\geq 0})^2} \frac{q^{\vec{m}^t\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)\vec{m}-\frac{1}{42}}}{(q)_{\vec{m}}} \tag{3.4.29}$$

and $\mathcal{M}(8,3)$ has characters

$$\chi_{1,1}^{8,3} = \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^2\\ m_2\equiv 0 \pmod 2}} \frac{q^{\frac{1}{2}\vec{m}^t\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)\vec{m}+\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^t\vec{m}-\frac{7}{32}}}{(q)_{\vec{m}}} \tag{3.4.30}$$

$$\chi_{1,2}^{8,3} = \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^2\\ m_1\equiv 0 \pmod 2}} \frac{q^{\frac{1}{2}\vec{m}^t\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)\vec{m}+\frac{1}{2}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)^t\vec{m}}}{(q)_{\vec{m}}} \tag{3.4.31}$$

$$\chi_{1,3}^{8,3} = \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^2\\ m_2\equiv 0 \pmod 2}} \frac{q^{\frac{1}{2}\vec{m}^t\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)\vec{m}-\frac{1}{32}}}{(q)_{\vec{m}}} \tag{3.4.32}$$

$$\chi_{1,4}^{8,3} = \sum_{\vec{m}\in(\mathbb{Z}_{\geq 0})^2} \frac{q^{\frac{1}{2}\vec{m}^t\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)\vec{m}+\frac{1}{2}\left(\begin{smallmatrix}2\\1\end{smallmatrix}\right)^t\vec{m}+\frac{1}{8}}}{(q)_{\vec{m}}} \tag{3.4.33}$$

$$\chi_{1,5}^{8,3} = \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^2\\ m_2\equiv 1 \pmod 2}} \frac{q^{\frac{1}{2}\vec{m}^t\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)\vec{m}-\frac{1}{32}}}{(q)_{\vec{m}}} \tag{3.4.34}$$

$$\chi_{2,2}^{8,3} = \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^2\\ m_1\equiv 1 \pmod 2}} \frac{q^{\frac{1}{2}\vec{m}^t\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)\vec{m}+\frac{1}{2}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)^t\vec{m}}}{(q)_{\vec{m}}} \tag{3.4.35}$$

$$\chi_{2,1}^{8,3} = \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^2\\ m_2\equiv 1 \pmod 2}} \frac{q^{\frac{1}{2}\vec{m}^t\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)\vec{m}+\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^t\vec{m}-\frac{41}{32}}}{(q)_{\vec{m}}}\ . \tag{3.4.36}$$

One can observe something which appears to hold for any conformal field theory for which fermionic character expressions are known: $\vec{b}$ is zero for the character corresponding to the smallest of the dimensions of the primary fields, $h_{\min}$. $A = \frac{2}{3}\left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)$ is related to the $\mathbb{Z}_3$ *parafermionic theory* [FZ85], $A = \frac{1}{2}\left(\begin{smallmatrix}3&2\\2&4\end{smallmatrix}\right)$ corresponds to $\mathcal{M}(7,3)$, and $A = \left(\begin{smallmatrix}4&1\\1&1\end{smallmatrix}\right)$ is known to correspond to the *tricritical Ising model* which has been identified with the minimal model $\mathcal{M}(5,4)$ of effective central charge $c_{\text{eff}}^{5,4} = \frac{7}{10}$ [FQS85], although it is not known to correspond to the vacuum character $\chi_{1,1}^{5,4}$. But there is another way to express the vacuum character (and the others as well) of the minimal model $\mathcal{M}(5,4)$ in a fermionic way. Trevor A. Welsh quite recently gave instructions on how to compute the fermionic expressions for all characters of all minimal models $\mathcal{M}(p,p')$ [Wel05]. These instructions however, are quite lengthy and computing the fermionic character expressions can be a very tedious task, though in principle not very difficult. The construction is based on continuous fractions and on tools like lattice paths, finitized characters, Takahashi trees and lengths and is an extension of the work of Alexander Berkovich, Barry McCoy and Anne Schilling [BMS98]. However, most of the results will not be of the form given in Nahm's conjecture, but will just be a linear combination of fundamental fermionic forms. For example, the vacuum character for the minimal model $\mathcal{M}(5,4)$ according to Welsh would be

$$\chi_{1,1}^{5,4} = \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^2 \\ m_1,m_2\equiv 0 \pmod 2}} \frac{q^{\frac{1}{4}\vec{m}^t\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}\vec{m}-\frac{7}{240}}}{(q)_{m_1}} \begin{bmatrix} \frac{m_1}{2} \\ m_2 \end{bmatrix}_q , \tag{3.4.37}$$

which is a special case of the general fermionic expression for unitary minimal model $\mathcal{M}(p+1,p)$ characters

$$q^{\alpha}\chi_{1,1}^{p+1,p} = \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^{p-2} \\ m_i \text{ even}}} \frac{q^{\frac{1}{4}\vec{m}^t C_{A_{p-2}}\vec{m}}}{(q)_{m_1}} \prod_{i=2}^{p-2} \begin{bmatrix} ((\mathbb{1}-\frac{1}{2}C_{A_{p-2}})\vec{m})_i \\ m_i \end{bmatrix}_q . \tag{3.4.38}$$

This can be computed by the methods of Welsh for all possible combinations of $r$ and $s$, but for simplicity, only the vacuum character is given here. In general, a 'finite'[5] $q$-binomial coefficient will always occur in a fermionic character expression of a minimal model $\mathcal{M}(p,p')$ when

$$\sum_{i=1}^{n} c_i > 3 , \tag{3.4.39}$$

where the $c_i$ are determined by the continued fraction

$$\frac{p}{p'} = c_0 + \cfrac{1}{c_1 + \cfrac{1}{\ddots\, c_{n-2} + \cfrac{1}{c_{n-1} + \cfrac{1}{c_n}}}} . \tag{3.4.40}$$

In the case of $\mathcal{M}(5,4)$, $c_1 = 4$, so the fermionic character expressions according to Welsh will contain q-binomial coefficients, the number of which being determined by the values of $p$, $p'$ and also $r$ and $s$. But since fermionic expressions for a character are not unique, i.e. there are different fermionic expressions for the same character, there may also be simpler fermionic expressions for a given character out there than those given in [Wel05]. It is conjectured that

[5]Note that $\lim_{N'\to\infty}\begin{bmatrix} N'+M \\ M \end{bmatrix}_q = \frac{1}{(q)_M}$.

the fact that there are different fermionic representations for the same character may point to the various integrable massive perturbations of the conformal field theory [KM90, KM92].

For example, $A = \left(\begin{smallmatrix}4&1\\1&1\end{smallmatrix}\right)$ corresponds to $\mathcal{M}(5,4)$, as mentioned above. Although it is not known to correspond to the vacuum character[6] $\chi_{1,1}^{5,4}$, it is known [Byt99a] to correspond to two other characters of this model, namely $\chi_{2,1}^{5,4}$ and $\chi_{2,2}^{5,4}$. The latter two characters have at least four fermionic representations:

$$\chi_{2,2}^{5,4} = \sum_{\vec{m}\in(\mathbb{Z}_{\geq 0})^2} \frac{q^{\frac{1}{4}\vec{m}^t\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)\vec{m}+\frac{1}{2}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)^t\vec{m}+\frac{1}{120}}}{(q)_{\vec{m}}} \tag{3.4.41}$$

$$= \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^2\\ m_1\equiv 1 \pmod 2\\ m_2\equiv 0 \pmod 2}} \frac{q^{\frac{1}{4}\vec{m}^t\left(\begin{smallmatrix}2&-1\\-1&2\end{smallmatrix}\right)\vec{m}-\frac{1}{2}\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)^t\vec{m}+\frac{1}{120}}}{(q)_{m_1}} \begin{bmatrix}\frac{m_1+1}{2}\\ m_2\end{bmatrix}_q \tag{3.4.42}$$

$$= \sum_{\vec{m}\in(\mathbb{Z}_{\geq 0})^2} \frac{q^{\frac{1}{4}\vec{m}^t\left(\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right)\vec{m}+\frac{1}{2}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)^t\vec{m}+\frac{1}{120}}}{(q^2)_{m_1}(q)_{m_2}} \tag{3.4.43}$$

$$= \sum_{\vec{m}\in(\mathbb{Z}_{\geq 0})^3} \frac{(-1)^{m_3} q^{\frac{1}{2}\vec{m}^t\left(\begin{smallmatrix}1&0&0\\0&1&1\\0&1&1\end{smallmatrix}\right)\vec{m}+\frac{1}{2}\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)^t\vec{m}+\frac{1}{120}}}{(q)_{\vec{m}}} \tag{3.4.44}$$

and similarly for $\chi_{2,1}^{5,4}$, although the last two expressions can be called fermionic only in a broader sense, sometimes referred to as *anyonic* interpretation of Virasoro characters [BF98]. Since only two characters have been found in this way, there may exist another model with $c_{\text{eff}} = \frac{7}{10}$ that shares some of the characters of $\mathcal{M}(5,4)$ and for which all characters including the vacuum character sport the matrix $A = \frac{1}{2}\left(\begin{smallmatrix}4&1\\1&1\end{smallmatrix}\right)$. Conversely,

$$\chi_{1,1}^{5,4} = \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^7\\ m_1+m_3+m_6\equiv 0 \pmod 2}} \frac{q^{\vec{m}^t C_{E_7}^{-1}\vec{m}-\frac{7}{240}}}{(q)_{\vec{m}}} \tag{3.4.45}$$

with

$$C_{E_7}^{-1} = \begin{pmatrix} \frac{3}{2} & 1 & \frac{3}{2} & 2 & 2 & \frac{5}{2} & 3\\ 1 & 2 & 2 & 2 & 3 & 3 & 4\\ \frac{3}{2} & 2 & \frac{7}{2} & 3 & 4 & \frac{9}{2} & 6\\ 2 & 2 & 3 & 4 & 4 & 5 & 6\\ 2 & 3 & 4 & 4 & 6 & 6 & 8\\ \frac{5}{2} & 3 & \frac{9}{2} & 5 & 6 & \frac{15}{2} & 9\\ 3 & 4 & 6 & 6 & 8 & 9 & 12 \end{pmatrix} \tag{3.4.46}$$

being the inverse Cartan matrix of the simply-laced Lie algebra $E_7$. However, only the vacuum character and $\chi_{3,1}^{5,4}$ are known to correspond to $C_{E_7}^{-1}$.

[6] $\mathcal{M}(5,4)$ does belong to the unitary series $\mathcal{M}(p+1,p)$ of minimal models, and thus $h_{\min} = 0$.

Even more peculiar is the following additional fermionic expression for the vacuum character of the Ising model discussed above, featuring the inverse Cartan matrix of $E_8$:

$$\chi_{1,1}^{4,3} = \sum_{\substack{m=0 \\ m\equiv 0 \pmod 2}}^{\infty} \frac{q^{\frac{m^2}{2}-\frac{1}{48}}}{(q)_m} \tag{3.4.47}$$

$$= \sum_{\vec{m}\in(\mathbb{Z}_{\geq 0})^8} \frac{q^{\vec{m}^t C_{E_8}^{-1}\vec{m}+\frac{1}{48}}}{(q)_{\vec{m}}} \tag{3.4.48}$$

with

$$C_{E_8}^{-1} = \begin{pmatrix} 2 & 2 & 3 & 3 & 4 & 4 & 5 & 6 \\ 2 & 4 & 4 & 5 & 6 & 7 & 8 & 10 \\ 3 & 4 & 6 & 6 & 8 & 8 & 10 & 12 \\ 3 & 5 & 6 & 8 & 9 & 10 & 12 & 15 \\ 4 & 6 & 8 & 9 & 12 & 12 & 15 & 18 \\ 4 & 7 & 8 & 10 & 12 & 14 & 16 & 20 \\ 5 & 8 & 10 & 12 & 15 & 16 & 20 & 24 \\ 6 & 10 & 12 & 15 & 18 & 20 & 24 & 30 \end{pmatrix} . \tag{3.4.49}$$

When we discuss the quasi-particle interpretation of the fermionic character expressions, this will become interesting. However, this is the only explicit fermionic expression for any of the $\mathcal{M}(4,3)$ characters that is known corresponding to $E_8$, albeit there are linear combinations of Ising characters which can be represented in this form. A proof of the expression in terms of $E_8$ is given in [WP94].

The key to this is that the last two expressions belong to the coset construction models $\frac{(G_r^{(1)})_1\times(G_r^{(1)})_1}{(G_r^{(1)})_2}$ [GKO85, KKMM93a], some of which being isomorphic to minimal models, where $G_r$ is a simply-laced Lie algebra of rank r. The next section will provide detailed information on the $ADET$ related cases.

In general, the known cases indicate that for a fermionic character expression which includes finite $q$-binomial coefficients, there exists another fermionic expression which is of Nahm type (3.3.1), i.e. without finite $q$-binomial coefficients, and which sports a matrix of a higher rank.

## 3.5. $ADET$ Classification

The possibility of classifying fermionic character expressions according to simple Lie algebras is investigated further in this section.

All possible simple Lie algebras have been classified by Eugene Borisovich Dynkin. Geometric constraints imply that there are only four infinite families or five exceptional cases. The infinite families are labeled by $A_n$, $B_n$, $C_n$ or $D_n$ and the exceptional cases by $G_2$, $F_4$, $E_6$, $E_7$ and $E_8$, where $n$ is the number of nodes of the corresponding *Dynkin diagram*. Each of the above Lie algebras is assigned a Dynkin diagram, and the set of Dynkin diagrams is in one-to-one correspondence with the set of *Cartan matrices*. If one labels the nodes of a Dynkin diagram by $a \in \{1, 2, \ldots, n\}$, one can construct its Cartan matrix by setting $(C_{X_n})_{ab}(C_{X_n})_{ba}$ to the number of connecting lines between the nodes $a$ and $b$ of the Dynkin

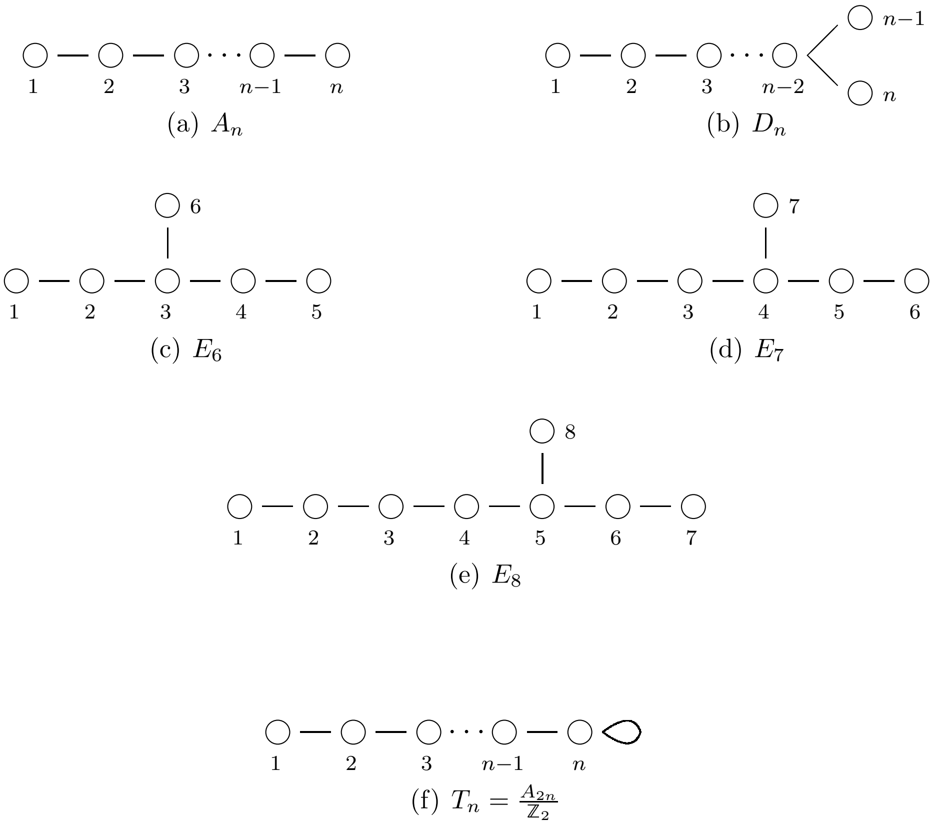



**Figure 3.1.:** (a)-(e): Dynkin diagrams for simple Lie algebras; (f): The 'tadpole' diagram

diagram to the Lie algebra $X_n$ and demanding that $(C_{X_n})_{ab} \leq 0$ and integer, and furthermore $(C_{X_n})_{aa} = 2$.

The $ADE$ graphs play an important role in many places in mathematics and physics. In conformal field theory, for example, they can be used to classify modular invariant partition functions [CIZ87b] and, furthermore, the Cartan matrices also appear in the quadratic form in the exponent of the fermionic character expressions.

In the following, the conformal field theories whose fermionic character expressions correspond to the $ADE$ graphs are reported as well as the additional artificial series of so-called *tadpole graphs*, which also appear in fermionic expressions. The corresponding Dynkin diagrams are displayed in figure 3.1. When the matrix $A$ in the quadratic form is mentioned, the reader is always referred to (3.2.6).[7]

**The $A_n$ series** corresponds to the unitary $\mathbb{Z}_{n+1}$ parafermionic theories with central charge $c_n = c^n_{\text{eff}} = \frac{2n}{n+3}$ [FZ85, FL88]. James Lepowsky and Mirko Primc in 1985 [LW81a,

[7]Upon comparing (3.2.6) with (3.3.1), one notices that the exponents differ by a factor of $\frac{1}{2}$ in the definition of the matrix $A$. Strictly speaking, one should write $\frac{1}{2}A$ in the exponent of every fermionic form, since in general $A = C_{X_r} \otimes C^{-1}_{Y_s}$ for some $X_r, Y_s \in \{A, D, E, T\}$ (as discussed in section 3.3) and in most cases occuring in this thesis $X_r = A_1$ and thus $A = 2C^{-1}_{Y_s}$. Therefore, since in most cases the factors 2 and $\frac{1}{2}$ cancel, the general fundamental fermionic form (3.2.6) is referred to in this thesis except in the single section 3.3 and where explicitly stated.

LP85] found fermionic expressions with sum restrictions for the $\mathbb{Z}_{n+1}$ characters, the latter consisting of the $A_1$ string functions of level $k$ by Kač and Peterson [KP84].

Moreover, $A_1$ corresponds to characters of the Ising model, namely (3.4.17) and (3.4.18) and $A_2$ corresponds to the characters of $\mathcal{M}(6,5)$, namely

$$\chi_{1,1}^{6,5} + \chi_{1,5}^{6,5} = \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^2 \\ m_1+2m_2\equiv 0 \pmod 3}} \frac{q^{\vec{m}^t C_{A_2}^{-1}\vec{m}-\frac{1}{30}}}{(q)_{\vec{m}}} \tag{3.5.1}$$

and

$$\chi_{1,3}^{6,5} = \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^2 \\ m_1+2m_2\equiv a \pmod 3}} \frac{q^{\vec{m}^t C_{A_2}^{-1}\vec{m}-\frac{1}{30}}}{(q)_{\vec{m}}}\ , \quad a \in \{-1,1\}\ . \tag{3.5.2}$$

Additionally, via $A = C_{A_n} \otimes C_{T_1}^{-1} = C_{A_n}$, this series also corresponds to the unitary series of minimal models (3.4.38).

**The $D_n$ series** corresponds to the unitary theory of a free boson compactified on a torus of radius $R = \sqrt{\frac{n}{2}}$ with central charge $c = c_{\text{eff}} = 1$. This theory has characters $\frac{\Theta_{\lambda,k}}{\eta}$ for $\lambda \in \{-k+1, \dots, k\}$ with $\lambda = 0$ denoting the vacuum character. The fermionic expressions for these characters can be all be written with the inverse Cartan matrix of $D_n$ in the quadratic form. We will discuss this later on, when we derive the fermionic expressions for the $c_{p,1}$ series of logarithmic conformal field theories, where we will see that the whole $c_{p,1}$ series corresponds to the $D_n$ series, i.e. the quadratic form in the fermionic character expressions is

$$\vec{m}^t C_{D_p}^{-1}\vec{m} \tag{3.5.3}$$

for all characters of the $c_{p,1}$ model. The sum restrictions state that the sum $m_{n-1}+m_n$ has to be either even or odd, depending on the chosen character of the model. For the subset of characters that are of the form $\frac{\Theta_{\lambda,k}}{\eta}$, both restrictions admit a realization. Note furthermore that due to the coincidence $D_3 = A_3$, some character functions corresponding to these two series are related.

**The exceptional algebra** $E_6$ corresponds to the unitary minimal model $\mathcal{M}(7,6)$ that is the *tricritical three-state Potts model* [FZ87] with central charge $c = \frac{6}{7}$, namely

$$\chi_{1,1}^{7,6} + \chi_{5,1}^{7,6} = \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^6 \\ m_1-m_2+m_4-m_5\equiv 0 \pmod 3}} \frac{q^{\vec{m}^t C_{E_6}^{-1}\vec{m}-\frac{c}{24}}}{(q)_{\vec{m}}} \tag{3.5.4}$$

and

$$\chi_{3,1}^{7,6} = \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^6 \\ m_1-m_2+m_4-m_5\equiv a \pmod 3}} \frac{q^{\vec{m}^t C_{E_6}^{-1}\vec{m}-\frac{c}{24}}}{(q)_{\vec{m}}}\ , \quad a \in \{-1,1\} \tag{3.5.5}$$

with

$$C_{E_6}^{-1} = \begin{pmatrix} \frac{4}{3} & \frac{2}{3} & 1 & \frac{4}{3} & \frac{5}{3} & 2 \\ \frac{2}{3} & \frac{4}{3} & 1 & \frac{5}{3} & \frac{4}{3} & 2 \\ 1 & 1 & 2 & 2 & 2 & 3 \\ \frac{4}{3} & \frac{5}{3} & 2 & \frac{10}{3} & \frac{8}{3} & 4 \\ \frac{5}{3} & \frac{4}{3} & 2 & \frac{8}{3} & \frac{10}{3} & 4 \\ 2 & 2 & 3 & 4 & 4 & 6 \end{pmatrix} . \tag{3.5.6}$$

By adding a suitable vector $\vec{b}$ to the exponent in the fermionic expression or by changing the sum restrictions, the other characters of $\mathcal{M}(7,6)$ might also be found to have fermionic representations of this type, but so far, none are known.

**The exceptional algebra** $E_7$ corresponds to the tricritical Ising unitary model $\mathcal{M}(5,4)$ with central charge $c = \frac{7}{10}$ mentioned in the previous section. Here,

$$\chi_{1,1}^{5,4} = \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^7 \\ m_1+m_3+m_6\equiv 0 \pmod 2}} \frac{q^{\vec{m}^t C_{E_7}^{-1}\vec{m} - \frac{c}{24}}}{(q)_{\vec{m}}} \tag{3.5.7}$$

and

$$\chi_{3,1}^{5,4} = \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^7 \\ m_1+m_3+m_6\equiv 1 \pmod 2}} \frac{q^{\vec{m}^t C_{E_7}^{-1}\vec{m} - \frac{c}{24}}}{(q)_{\vec{m}}} . \tag{3.5.8}$$

**The exceptional algebra** $E_8$ corresponds to the Ising model $\mathcal{M}(4,3)$ and, as also mentioned in the previous section,

$$\chi_{1,1}^{4,3} = \sum_{\vec{m}\in(\mathbb{Z}_{\geq 0})^8} \frac{q^{\vec{m}^t C_{E_8}^{-1}\vec{m} - \frac{1}{48}}}{(q)_{\vec{m}}} . \tag{3.5.9}$$

**The** $T_n$ **series** , often called $\frac{A_{2n}}{\mathbb{Z}_2}$, corresponds to the series of non-unitary Virasoro minimal models $\mathcal{M}(2n+3,2)$ with effective central charge $c_{\text{eff}}^k = \frac{2n}{2n+3}$. Their characters admit a product form [ABS90, FNO92], which is one side of the Andrews-Gordon identities [Gor61, And74, Bre80, And84] (see also appendix A)

$$\sum_{m_1,\ldots,m_n=0}^{\infty} \frac{q^{M_1^2+\ldots+M_n^2+M_a+\ldots+M_n}}{(q)_{m_1}\cdots(q)_{m_n}} = \prod_{\substack{m\not\equiv 0 \pmod{2n+3} \\ m\not\equiv \pm a \pmod{2n+3}}} (1-q^m)^{-1} \tag{3.5.10}$$

with $M_k := m_1+\ldots+m_k$. Basil Gordon gave the combinatorial and George E. Andrews the analytical proof. The other side consists of the fermionic sum representations for the characters of $\mathcal{M}(2n+3,2)$. This original formulation can be rewritten in order to match the fermionic forms as

$$\chi_{1,j}^{2n+3,2}(q) = q^{h_{1,j}^{2n+3,2} - \frac{c_{2n+3,2}}{24}} \sum_{\vec{m}\in(\mathbb{Z}_{\geq 0})^n} \frac{q^{\vec{m}^t C_{T_n}^{-1}\vec{m} + \vec{b}_{T_n}^t \vec{m}}}{(q)_{\vec{m}}} \tag{3.5.11}$$

with $C_{T_n}$ being the Cartan matrix of the tadpole graph which differs from $C_{A_n}$ only by a 1 instead of a 2 in the component $(C_{T_n})_{nn}$ and $\vec{b}^t_{T_n} = (\underbrace{0,\ldots,0}_{n \text{ times}}, 1, 2, \ldots, k-n)$.
Note that the Andrews-Gordon identities reduce to the Rogers-Ramanujan identities for $n = 1$ and $a \in \{1, 2\}$, i.e. for $\mathcal{M}(5, 2)$.

Most of these expressions were found and verified by using Mathematica and explicit proofs were lacking for most of them [KKMM93a]. But that situation changed during the following years. A particular example is the fermionic character expression for $\chi^{4,3}_{1,1}$ related to $E_8$, for which S. Ole Warnaar and Paul A. Pearce found a proof based on the so-called dilute $A_3$ model [WP94]. A different direction that allowed for many of the identities to be proven was found by Melzer [Mel94]. He observed that Virasoro characters have a natural *finitized* version in terms of so-called *path spaces* or *corner-transfer matrix sums* in the *rough solid-on-solid (RSOS) model* [ABF84]. This method of proving the identities has been extended in [Ber94] and references therein. The fact that there are different fermionic expressions for a single character (in the sense that the matrix $A$ in the quadratic form is different) is demonstrated impressively by the vacuum character $\chi^{4,3}_{1,1}$ of the Ising model. There is a sum representation related to $A_1$ and a sum representation related to $E_8$. Let us discuss this. In [KM90], Timothy Klassen and Ezer Melzer investigated integrable massive scattering theories. There, the $ADE$ algebras describe certain perturbations of coset conformal field theories [GKO85] related to $ADE$. These algebras are the same. For example, the energy perturbation of the Ising model, which is called *Ising field theory*, corresponds to $A_1$ and to the conformal limit of Kaufman's representation of the general Ising model in the absence of a magnetic field in terms of a single, free fermion [Kau49], while the magnetic perturbation corresponds to a scattering theory of eight different particle species [Zam89]. Later on in this thesis, when we demonstrate the quasi-particle interpretation of the fermionic character expressions, we will see that the $E_8$ character corresponds also to a system of eight quasi-particle species with exactly the charges in [Zam89] reproduced by the sum restrictions. This is another example that different fermionic expressions for the same character point to different integrable perturbations of the conformal field theory in consideration.

Furthermore, symmetries of the character $\chi^{p,p'}_{r,s}$ with repect to its parameters (see (3.4.11) and (3.4.12)) add to the non-uniqueness of a fermionic character expression. For instance,

$$\chi^{p,\alpha p'}_{\alpha r,s} = \chi^{\alpha p,p'}_{r,\alpha s} \quad \alpha \in \mathbb{Z}_{\geq 1} \ , \ \langle p, \alpha p' \rangle = \langle \alpha p, p' \rangle = 1 \tag{3.5.12}$$

implies that the characters of $\mathcal{M}(6, 5)$ are related to those of $\mathcal{M}(10, 3)$.

## 3.6. Characters of $SU(2)$ Level $k$ WZW Models

### 3.6.1. Bosonic Character Expressions

In this section, bosonic and fermionic character expressions for $SU(2)_k$ WZW models are displayed and new fermionic expressions are given, based on our recent article [FGK07]. The original bosonic expressions have been obtained by Kač and Peterson [KP84] for the characters of the affine Lie algebra $A^{(1)}_1$ for integrable representations (integer or half-integer spin $l \leq \frac{k}{2}$) (in short, affine characters) of level $k$ and spin $l$ as

$$\chi^{\mathrm{aff}}_{\lambda,k}(\tau) = \frac{1}{\eta^3(\tau)} \sum_{n=-\infty}^{\infty} (nN+\lambda) \mathrm{e}^{\mathrm{i}\pi\tau \frac{(nN+\lambda)^2}{N}} \tag{3.6.1}$$

with $N = 2(k+2)$ and $\lambda = 2l+1$, which can be rephrased in terms of the affine $\Theta$-function (cf. section 2.5) as

$$\chi^{\mathrm{aff}}_{\lambda,k-2} = \frac{(\partial\Theta)_{\lambda,k}}{\eta^3} \tag{3.6.2}$$

for $0 < \lambda < k$.

### 3.6.2. Fermionic Character Expressions from Spinon Bases

The $ADET$ pattern does not obviously appear to exhaust the spectrum of fermionic character expressions. For example, there is also a quite different source for fermionic character sums: It was observed that a 'discretization' [HHT$^+$92] of the $\widehat{su}(N)_1$ *Wess-Zumino-Witten (WZW) model* is connected to the $su(N)$ *Haldane-Shastry model* [Hal88, Sha88], the latter being the integrable model of a spin chain with long-range interaction, whose elementary excitations can be described in terms of *spinons*, free particles obeying *fractional statistics*, i.e. the fundamental excitations (quasi-particles) over a many-body ground state carry quantum numbers which are fractions of the quantum numbers carried by the microscopic degrees of freedom in the system (e.g. Laughlin quasi-particles in the *fractional quantum Hall effect*, which have fractional charge [Lau83, TSG82, SGJE97]). In other words, the conformal limit of the $su(N)$ spin chain with long-range interaction is the $su(n)$ level one WZW model. (Most properties of the spin chain still apply to this limit.) This leads to a description of the basis of states of the WZW model in terms of spinons, and from these spinons, fermionic sum representations for the characters may be obtained [BPS94, BLS94b, BLS95a]. The quasi-particle description corresponding to fermionic sum representations for characters is investigated in detail in chapter 5. Based on this, Schoutens then proposed a very general method for investigating the exclusion statistics of quasi-particles in conformal field theory spectra [Sch97] (see also [BS99]), employing recursion relations for truncations of the chiral conformal field theories spectrum. This approach includes Haldane's *fractional exclusion statistics* in special cases: The exclusion statistics of CFT quasi-particles obtained from the recursion method agree with the fractional exclusion statistics by Haldane. Then, the quasi-particle character formulae take the form of the fermionic sum representations. This correspondence between fermionic sum representations of characters and Haldane's statistics was discussed in [BM98, Sch99].

In this section, we are concerned about fermionic character expressions for the irreducible integrable representations of $A_1^{(1)}$ at level $k-2$. A spinon basis for the $\mathfrak{su}(2)_k$ spectrum was constructed in [BLS95a] and the corresponding statistics were described in [FS98].

The Haldane-Shastry long-range spin chain is integrable and has *Yangian symmetry*, as well as the $SU(2)$ level one WZW model. The chiral symmetry algebra of the latter is $\widehat{s\ell_2}$, the affine Lie algebra corresponding to $s\ell_2$. The Yangian $Y(s\ell_2)$ is another, highly non-trivial symmetry structure of the theory, which is natural in a spinon formulation. This algebra can be represented on the Hilbert space of the $SU(2)_1$ WZW model. While the description of that Hilbert space in terms of $\widehat{s\ell_2}$ is difficult due to the existence of singular vectors, it was found to be simple in terms of the Yangian symmetry algebra: It can be constructed as a Fock space of massless spinons satisfying generalized commutation relations. From this, fermionic quasi-particle sum representations for the characters have been derived [BLS94a]. In [BLS95a], this procedure has been generalized to levels greater than one. A spinon basis for the $\mathfrak{su}(2)_k$ spectrum was proposed and fermionic character expressions were obtained, which have been verified to high order. However, these fermionic formulae are not of fundamental fermionic form type (cf. (3.2.6)). In particular, they consist of finite $q$-binomial coefficients. But for special cases, there are different fermionic expressions which are of fundamental fermionic form type [FS93]. We display at first the known fermionic expressions of both types and then present new fermionic expressions of fundamental fermionic form type below, based on our recent article [FGK07]. The spinon Fock space of the $SU(2)$ WZW model at level $k$ decomposes into a direct sum of integrable highest weight modules of $(\widehat{s\ell_2})_k$. The fermionic character expressions for the irreducible integrable representations of $A_1^{(1)} = \widehat{s\ell_2}$ at level $k-2$ [BLS95a] are given by

$$\chi^{\text{aff}}_{\lambda,k-2} = \frac{(\partial\Theta)_{\lambda,k}(\tau)}{\eta^3(\tau)} = \sum_{\substack{m_1,\dots,m_{k-1}=0 \\ (\vec{m}')_i \equiv (\vec{Q}(\lambda))_i \pmod 2}}^{\infty} \frac{q^{\vec{m}^t B_k \vec{m} + c^\sharp_{\lambda,k}}}{(q)_{m_1}(q)_{m_2}} \prod_{i=3}^{k-1} \begin{bmatrix} \lceil (\frac{1}{2}(2-C_{A_{k-2}})\vec{m}')_{i-1} \rceil \\ m_i \end{bmatrix}_q \tag{3.6.3}$$

for $0 < \lambda < k$ with $\vec{m}'^t = (m_1 + m_2, m_3, m_4, \dots, m_{k-1})$ and

$$4B_k = C_k + C_{A_{k-1}} \,, \quad (C_k)_{ij} = \begin{cases} -1 & \text{if } \ i+j \ \text{ is even and } \ i+j \le 4 \\ 2 & \text{if } \ i+j \ \text{ is odd and } \ i+j \le 4 \\ 0 & \text{if } \ i+j > 4 \end{cases} \,, \tag{3.6.4}$$

where $C_{A_k}$ is the Cartan matrix of the Lie algebra $A_k \cong s\ell_{k+1}$ and $c^\sharp_{\lambda,k} = \frac{2\lambda^2+k-2k\lambda}{8k}$. Given any $x \in \mathbb{R}$, $\lceil x \rceil$ and $\lfloor x \rfloor$ mean the next integer greater than or equal to $x$ and the next integer less than or equal to $x$, respectively. The following restrictions hold for the sum variables: $(\vec{m}')_i = (\vec{Q}(\lambda))_i \pmod 2$ with $\vec{Q}(\lambda) = ((\sum_{j=0}^{\lfloor \frac{\lambda}{2}-1 \rfloor} \delta_{i,\lambda-(2j+1)})_i : i \in \{1,\dots,k-2\}) \in (\mathbb{Z}_2)^{k-2}$, i.e. $\vec{Q}(\lambda)$ is either of the form $(1,0,1,0,\dots,1,0,0,0,\dots,0)$ if $\lambda$ is odd or of the form $(0,1,0,1,\dots,1,0,0,0\dots,0)$ if $\lambda$ is even.[8] We see from (3.6.3) that the module at level zero is trivial, its character being

$$\frac{(\partial\Theta)_{1,2}(\tau)}{\eta(\tau)^3} = 1 \,. \tag{3.6.5}$$

For $\lambda = 1$ and $\lambda = k-1$, there exists another expression. In both cases, it consists of a single fundamental fermionic form without sum restrictions and has $2(k-2)$ different sum indices.

The expression for $\lambda = 1$ was given without proof (except for level 1, see below) by Feigin and Stoyanovsky using a flag manifold approach [FS93] and reads

$$\frac{(\partial\Theta)_{1,k}(\tau)}{\eta^3(\tau)} = \sum_{\vec{m} \in (\mathbb{Z}_{\ge 0})^{2(k-2)}} \frac{q^{\frac{1}{2}\vec{m}^t (C_{A_2} \otimes C^{-1}_{T_{k-2}})\vec{m} + c^\flat_{1,k}}}{(q)_{\vec{m}}} \,, \tag{3.6.6}$$

[8] The number and the placement of entries 1 in the latter vector may be changed in certain ways, but then an inner product $\vec{b}^t\vec{m}$ with the $k-1$-component vector $\vec{b}^t = (\frac{1}{2}, -\frac{1}{2}, 0, \dots, 0)$ has to be added to the quadratic form in the numerator of (3.6.3).

where $C_{A_2}$ is as above, $C_{T_k}^{-1}$ is the inverse of the $k \times k$ Cartan matrix of the tadpole graph[9] and the constant $c^\flat_{\lambda,k} = \frac{\lambda^2}{4k} - \frac{1}{8}$.

For $\lambda = k-1$, we present in [FGK07] the following fermionic expression:

$$\frac{(\partial\Theta)_{k-1,k}(\tau)}{\eta^3(\tau)} = \sum_{\vec{m}\in(\mathbb{Z}_{\geq 0})^{2(k-2)}} \frac{q^{\frac{1}{2}\vec{m}^t(C_{A_2}\otimes C^{-1}_{T_{k-2}})\vec{m}+(\vec{a}_2\otimes\vec{b}_{k-2})^t\vec{m}+c^\flat_{k-1,k}}}{(q)_{\vec{m}}} \tag{3.6.7}$$

with $\vec{a}_2^t = (1,-1)$ and $\vec{b}_k^t = (1,2,3,\ldots,k)$. It has been checked numerically up to $k=4$ and high order and is assumed to hold for higher values of $k$.

For example,

$$\frac{(\partial\Theta)_{3,4}(\tau)}{\eta^3(\tau)} = \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^3 \\ m_1+m_2\equiv 0 \pmod 2 \\ m_3\equiv 1 \pmod 2}} \frac{q^{\frac{1}{4}\vec{m}^t\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 2\end{pmatrix}\vec{m}-\frac{1}{16}}}{\prod_{i=1}^2 (q)_{m_i}} \begin{bmatrix} \frac{m_1+m_2}{2} \\ m_3 \end{bmatrix}_q \tag{3.6.8}$$

or, equivalently,

$$\frac{(\partial\Theta)_{3,4}(\tau)}{\eta^3(\tau)} = \sum_{\vec{m}\in(\mathbb{Z}_{\geq 0})^4} \frac{q^{\frac{1}{2}\vec{m}^t\begin{pmatrix} 2 & -1 & 2 & -1 \\ -1 & 2 & -1 & 2 \\ 2 & -1 & 4 & -2 \\ -1 & 2 & -2 & 4\end{pmatrix}\vec{m}+\begin{pmatrix} 1 \\ -1 \\ 2 \\ -2\end{pmatrix}^t\vec{m}+\frac{17}{48}}}{(q)_{\vec{m}}}\ . \tag{3.6.9}$$

For level 1, i.e. $k=3$, both types of fermionic expressions, (3.6.3) and (3.6.6), can be shown to be equivalent:

$$\frac{(\partial\Theta)_{1,3}(\tau)}{\eta^3(\tau)} = \sum_{\vec{m}=0}^{\infty} \frac{q^{\frac{1}{2}\vec{m}^t\left(\begin{smallmatrix} 2 & -1 \\ -1 & 2\end{smallmatrix}\right)\vec{m}-\frac{1}{24}}}{(q)_{\vec{m}}} \tag{3.6.10}$$

$$= \sum_{m_1,m_2\geq 0}^{\infty} \frac{q^{m_1^2+m_2^2-m_1m_2-\frac{1}{24}}}{(q)_{m_1}(q)_{m_2}} = q^{-\frac{1}{24}} \sum_{m\in\mathbb{Z}} q^{m^2} \sum_{\substack{m_1,m_2\geq 0 \\ m_1-m_2=m}} \frac{q^{m_1m_2}}{(q)_{m_1}(q)_{m_2}} \tag{3.6.11}$$

$$= \sum_{m\in\mathbb{Z}} \frac{q^{m^2}}{\eta(q)} = \frac{\Theta_{0,1}(q)}{\eta(q)} \tag{3.6.12}$$

$$= \sum_{\substack{\vec{m} \\ m_1+m_2\equiv 0 \pmod 2}}^{\infty} \frac{q^{\frac{1}{4}\vec{m}^t\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1\end{smallmatrix}\right)\vec{m}}}{(q)_{\vec{m}}}\ , \tag{3.6.13}$$

where for the fourth equality we have made use of the Durfee rectangle identity (3.7.15) (see e.g. [And84]) and for the last equality, we used a relation that will be proven in 3.7.2.

[9] The $C_{T_k}$ Cartan matrix differs from the $C_{A_k}$ Cartan matrix only by a 1 instead of a 2 in the lower right corner.

# 3.7. Characters of the Triplet Algebras $\mathcal{W}(2, 2p-1, 2p-1, 2p-1)$

## 3.7.1. Characters in Bosonic Form

The triplet $\mathcal{W}$ algebras are rational conformal field theories[10][GK96, Flo96, CF06], i.e. the number of highest weight representations of the $\mathcal{W}$-algebra is finite, and the generalized character functions span a finite dimensional representation of the modular group. Knowing the vacuum character is sufficient in proving rationality of the theory.

One can calculate the $\mathcal{W}$ character of the vacuum representation by summing up all the Virasoro characters of the highest weight representations corresponding to integer values of $h$, the latter being given by

$$h_{2k+1,1} = k^2 p + kp - k \ . \qquad (3.7.1)$$

All the corresponding primary fields belong to degenerate conformal families. By means of a standard free-field construction [BPZ84, DF84, DF85a, DF85b], it turns out that the representations with these highest weights $h_{2k+1,1}$ correspond to a set of relatively local chiral vertex operators $\varPhi_{2k+1,1}$. It follows that the local chiral algebra can be extended by them. The conditions for the existence of well-defined chiral vertex operators [Kau95, Kau00] result in abstract fusion rules which imply that the local chiral algebra generated by only the stress-energy tensor and the field $\varPhi_{3,1}$ closes. Repeated application of the so-called *screening charge operator* $Q$ [Fel89] on $\varPhi_{3,1}$ generates a multiplet structure. Thus, one also has to take care of the $\mathfrak{su}(2)$ symmetry of the triplet of fields, which results in the multiplicity of the Virasoro representation $|h_{2k+1,1}\rangle$ being $2k+1$. E.g., since $h_{3,1} = 2p-1$ and its multiplicity is three, it matches the fact that we have a triplet of fields in the algebra $\mathcal{W}(2, 2p-1, 2p-1, 2p-1)$. The vacuum representation of the $\mathcal{W}$-algebra can then be written as the following decomposition of the Hilbert space:

$$\mathcal{H}_{|0\rangle} = \bigoplus_{k \in \mathbb{Z}} (2k+1) \mathcal{H}^{\mathrm{Vir}}_{|h_{2k+1,1}\rangle} \ . \qquad (3.7.2)$$

The embedding structure of Feigin and Fuks [FF83] (see section 1.2.5) in the case of $p' = 1$ implies that the Virasoro characters corresponding to $h_{2k+1,1}$ – these are the only integer-valued for all $p$ – are given by

$$\chi^{\mathrm{Vir}}_{2k+1,1} = \frac{q^{\frac{1-c_{p,1}}{24}}}{\eta(q)} \left( q^{h_{2k+1,1}} - q^{h_{2k+1,-1}} \right) \ . \qquad (3.7.3)$$

[10] They are rational in the generalized sense discussed in section 1.4.3, since indecomposable representations occur.

It is thus possible to compute the vacuum character as

$$\begin{aligned}
\chi_0^{\mathcal{W}}(q) &= \sum_{k\in\mathbb{Z}_{\geq 0}} (2k+1)\chi^{\mathrm{Vir}}_{2k+1,1}(q) \\
&= \frac{q^{\frac{1-c_{p,1}}{24}}}{\eta(q)} \left( \sum_{k\in\mathbb{Z}_{\geq 0}} (2k+1) q^{h_{2k+1,1}} - \sum_{k\in\mathbb{Z}_{\geq 0}} (2k+1) q^{2k+1,-1} \right) \\
&= \frac{q^{\frac{(p-1)^2}{4p}}}{\eta(q)} \left( \sum_{k\in\mathbb{Z}_{\geq 0}} (2k+1) q^{h_{2k+1,1}} - \sum_{k\in\mathbb{Z}_{\geq 1}} (2k-1) q^{-2k+1,1} \right) \\
&= \frac{q^{\frac{(p-1)^2}{4p}}}{\eta(q)} \left( \sum_{k\in\mathbb{Z}_{\geq 0}} (2k+1) q^{h_{2k+1,1}} + \sum_{k\in\mathbb{Z}_{\leq 1}} (2k+1) q^{2k+1,1} \right) \\
&= \frac{q^{\frac{(p-1)^2}{4p}}}{\eta(q)} \left( \sum_{k\in\mathbb{Z}} (2k+1) q^{h_{2k+1,1}} \right) \\
&= \frac{1}{\eta(q)} \left( \sum_{k\in\mathbb{Z}} (2k+1) q^{pk^2+kp-k+\frac{(p-1)^2}{4p}} \right) \\
&= \frac{1}{p\eta(q)} \left( \sum_{k\in\mathbb{Z}} (2pk+p) q^{\frac{(2pk+(p-1))^2}{4p}} \right) \\
&= \frac{1}{p\eta(q)} \left( (\partial\Theta)_{p-1,p}(q) + \Theta_{p-1,1}(q) \right) \ ,
\end{aligned} \tag{3.7.4}$$

where the symmetry property $h_{r,s} = h_{-r,-s}$ has been used and the $\Theta$-functions as defined in 2.5. The $h$-values of a given $\mathcal{W}$-algebra can be calculated by use of the free-field construction, using Jacobi identities and null field constraints (cf. section 1.4.3). The corresponding characters may be calculated as follows: The so-called modular differential equation (see e.g. [Flo96]) may be used to compute as many terms of the $q$-expansion of the character as are necessary to unambigiously identify the corresponding function, because the requirement of that function to be a modular form implies strong restrictions on that function. It turns out that if we assume that $c_{3p,3} = c_{p,1}$ corresponds to a minimal model, which of course it doesn't since $3p$ and 3 are not coprime, it is possible to read the resulting $h$-values of the given $c_{p,1}$ theory off that enlarged Kač table.

$\frac{\Theta_{\lambda,k}(\tau)}{\eta(\tau)}$ is a modular form of weight zero with respect to the generators $\mathcal{T}: \ \tau \mapsto \tau+1$ and $\mathcal{S}: \ \tau \mapsto -\frac{1}{\tau}$ of the modular group $PSL(2,\mathbb{Z})$. But since $\frac{(\partial\Theta)_{\lambda,k}(\tau)}{\eta(\tau)}$ is a modular form of weight one with respect to $\mathcal{S}$ (cf. section 2.5), some of the above character functions are of inhomogeneous modular weight, thus leading to $S$-matrices with $\tau$-dependent coefficients. However, adding

$$(\nabla\Theta)_{\lambda,k}(\tau) = \frac{\log q}{2\pi\mathrm{i}} \sum_{n\in\mathbb{Z}} (2kn+\lambda) q^{\frac{(2kn+\lambda)^2}{4k}} \ , \tag{3.7.5}$$

one finds a closed finite-dimensional representation of the modular group with constant $S$-matrix coefficients.

After all, it turns out that a complete set of character functions for the $c_{p,1}$ models that is closed under modular transformations [Flo97] is given by:

$$\chi_{0,p} = \frac{\Theta_{0,p}}{\eta} \qquad \text{representation to} \qquad h^{p,1}_{1,p} \tag{3.7.6a}$$

$$\chi_{p,p} = \frac{\Theta_{p,p}}{\eta} \qquad h^{p,1}_{1,2p} \tag{3.7.6b}$$

$$\chi^+_{\lambda,p} = \frac{(p-\lambda)\Theta_{\lambda,p} + (\partial\Theta)_{\lambda,p}}{p\eta} \qquad h^{p,1}_{1,p-\lambda} \tag{3.7.6c}$$

$$\chi^-_{\lambda,p} = \frac{\lambda\Theta_{\lambda,p} - (\partial\Theta)_{\lambda,p}}{p\eta} \qquad h^{p,1}_{1,3p-\lambda} \tag{3.7.6d}$$

$$\tilde{\chi}^+_{\lambda,p} = \frac{\Theta_{\lambda,p} + \mathrm{i}\alpha\lambda(\nabla\Theta)_{\lambda,p}}{\eta} \qquad h^{p,1}_{1,p+\lambda} \tag{3.7.6e}$$

$$\tilde{\chi}^-_{\lambda,p} = \frac{\Theta_{\lambda,p} - \mathrm{i}\alpha(p-\lambda)(\nabla\Theta)_{\lambda,p}}{\eta} \qquad h^{p,1}_{1,p+\lambda} \tag{3.7.6f}$$

where $0 < \lambda < k$, $k = pp' = p$, $\lambda = pr - p's = pr - s$ and with the *Jacobi-Riemann $\Theta$-function* and the *affine $\Theta$-function* defined as in 2.5.

Note that (3.7.6e) and (3.7.6f) are not characters of representations in the usual sense. Actually, these are regularized character functions and the $\alpha$-dependent part has an interpretation as torus vacuum amplitudes [FG06]. In the limit $\alpha \to 0$, they become the characters of the full reducible but indecomposable representations.

### 3.7.2. Fermionic Character Expressions for $\mathcal{W}(2, 3, 3, 3)$

Fermionic sum representation for the $c_{p,1}$ models had not been found before. They are presented in this work and our corresponding publication [FGK07]. In this section, the fermionic formulae for the case of $p = 2$ are derived. The expressions for $p > 2$ will follow later.

In the case of $p = 2$, the bosonic characters read:

$$\chi^+_{1,2} = \frac{\Theta_{1,2} + (\partial\Theta)_{1,2}}{2\eta} \qquad \text{vacuum irrep } \mathcal{V}_0 \text{ to } h_{1,1} = 0 \tag{3.7.7a}$$

$$\chi_{0,2} = \frac{\Theta_{0,2}}{\eta} \qquad \text{irrep to } h_{1,2} = -\tfrac{1}{8} \tag{3.7.7b}$$

$$\chi_{1,2} = \frac{\Theta_{1,2}}{\eta} \qquad \text{indecomp. rep } \mathcal{R}_0(\supset \mathcal{V}_0) \text{ to } h_{1,3} = 0 \tag{3.7.7c}$$

$$\chi_{2,2} = \frac{\Theta_{2,2}}{\eta} \qquad \text{irrep to } h_{1,4} = \tfrac{3}{8} \tag{3.7.7d}$$

$$\chi^-_{1,2} = \frac{\Theta_{1,2} - (\partial\Theta)_{1,2}}{2\eta} \qquad \text{irrep to } h_{1,5} = 1. \tag{3.7.7e}$$

When $\alpha \to 0$, the general forms (3.7.6e) and (3.7.6f) lead to the character expression (3.7.7c) [Kau95, Flo97]. Actually, there exist two indecomposable representations, $\mathcal{R}_0$ and $\mathcal{R}_1$ (cf. section 1.4.3), which, however, are equivalent and thus share the same character.

Explicitly,

$$\begin{aligned}
q^{-\frac{1}{12}}\chi^{+}_{1,2} =& 1 + q^2 + 4q^3 + 5q^4 + 8q^5 + 10q^6 + 16q^7 + 22q^8 + 32q^9 \\
&+ 47q^{10} + 64q^{11} + 88q^{12} + 120q^{13} + 161q^{14} + 212q^{15} \\
&+ 282q^{16} + 368q^{17} + 480q^{18} + 620q^{19} + 798q^{20} + \dots
\end{aligned} \tag{3.7.8}$$

$$\begin{aligned}
q^{\frac{1}{24}}\chi_{0,2} =& 1 + q + 4q^2 + 5q^3 + 9q^4 + 13q^5 + 21q^6 + 29q^7 + 46q^8 + 62q^9 \\
&+ 90q^{10} + 122q^{11} + 171q^{12} + 227q^{13} + 311q^{14} + 408q^{15} \\
&+ 545q^{16} + 709q^{17} + 933q^{18} + 1198q^{19} + 1555q^{20} + \dots
\end{aligned} \tag{3.7.9}$$

$$\begin{aligned}
q^{-\frac{1}{12}}\chi_{1,2} =& 1 + 2q + 3q^2 + 6q^3 + 9q^4 + 14q^5 + 22q^6 + 32q^7 + 46q^8 + 66q^9 \\
&+ 93q^{10} + 128q^{11} + 176q^{12} + 238q^{13} + 319q^{14} + 426q^{15} \\
&+ 562q^{16} + 736q^{17} + 960q^{18} + 1242q^{19} + 1598q^{20} + \dots
\end{aligned} \tag{3.7.10}$$

$$\begin{aligned}
q^{-\frac{11}{24}}\chi_{2,2} =& 2 + 2q + 4q^2 + 6q^3 + 12q^4 + 16q^5 + 26q^6 + 36q^7 + 54q^8 + 74q^9 \\
&+ 106q^{10} + 142q^{11} + 200q^{12} + 264q^{13} + 358q^{14} + 470q^{15} \\
&+ 626q^{16} + 810q^{17} + 1062q^{18} + 1362q^{19} + 1760q^{20} + \dots
\end{aligned} \tag{3.7.11}$$

$$\begin{aligned}
q^{-\frac{1}{12}}\chi^{-}_{1,2} =& 2q + 2q^2 + 2q^3 + 4q^4 + 6q^5 + 12q^6 + 16q^7 + 24q^8 + 34q^9 \\
&+ 46q^{10} + 64q^{11} + 88q^{12} + 118q^{13} + 158q^{14} + 214q^{15} \\
&+ 280q^{16} + 368q^{17} + 480q^{18} + 622q^{19} + 800q^{20} + \dots\ .
\end{aligned} \tag{3.7.12}$$

In the following, the fermionic expressions for $\frac{\Theta_{\lambda,2}(\tau)}{\eta(\tau)}$, $0 \leq \lambda \leq 2$, are being calculated at first. In this case, the bosonic expressions can be straightforward transformed to the fermionic ones: At first,

$$\frac{\Theta_{\lambda,k}}{(q)_\infty} = \sum_{n=-\infty}^{+\infty} \frac{q^{\frac{(2kn+\lambda)^2}{4k}}}{(q)_\infty} \tag{3.7.13}$$

$$= \frac{1}{(q)_\infty}\left( q^{\frac{\lambda^2}{4k}} + \sum_{n=1}^{\infty} q^{\frac{(2kn-\lambda)^2}{4k}} + \sum_{n=1}^{\infty} q^{\frac{(2kn+\lambda)^2}{4k}} \right) . \tag{3.7.14}$$

Then, an identity

$$\sum_{n=0}^{\infty} \frac{q^{n^2+nk}}{(q)_n (q)_{n+k}} = \frac{1}{(q)_\infty} \tag{3.7.15}$$

that can be proven using Durfee squares or the $q$-analogue of Kummer's theorem (see e.g. [And84, pp. 21,28]) is employed to turn (3.7.13) into

$$\sum_{m=0}^{\infty} \frac{q^{m^2}}{(q)_m^2} + \sum_{n_1=1}^{\infty}\sum_{m_1=0}^{\infty} \frac{q^{m_1^2+m_1(2n_1)+\frac{(k(2n_1)-\lambda)^2}{4k}}}{(q)_{m_1}(q)_{m_1+2n_1}} + \sum_{n_2=1}^{\infty}\sum_{m_2=0}^{\infty} \frac{q^{m_2^2+m_2(2n_2)+\frac{(k(2n_2)+\lambda)^2}{4k}}}{(q)_{m_1}(q)_{m_1+2n_2}} . \tag{3.7.16}$$

Setting $n_1 = \frac{m_2-m_1}{2}$ and $n_2 = \frac{m_1-m_2}{2}$ leads to

$$\sum_{m=0}^{\infty} \frac{q^{m^2}}{(q)_m^2} + \sum_{\substack{0\leq m_1<m_2=0 \\ m_1+m_2\equiv 0 \pmod 2}}^{\infty} \sum_{\substack{0\leq m_2<m_1=0 \\ m_1+m_2\equiv 0 \pmod 2}}^{\infty} \frac{q^{\frac{k}{4}(m_1^2+m_2^2)+\frac{2-k}{2}m_1m_2+\frac{\lambda}{2}(m_1-m_2)+\frac{\lambda^2}{4k}}}{(q)_{m_1}(q)_{m_2}} . \tag{3.7.17}$$

On the other hand,

$$\begin{aligned}\frac{\Theta_{\lambda,k}}{(q)_\infty} &= \sum_{n=-\infty}^{+\infty} \frac{q^{\frac{(2kn+\lambda)^2}{4k}}}{(q)_\infty} \\ &= \sum_{n_1=1}^{\infty}\sum_{m_1=0}^{\infty} \frac{q^{m_1^2+m_1(2n_1-1)+\frac{-2k(k-\lambda)(2n_1-1)+(k-\lambda)^2+k^2(2n_1-1)^2}{4k}}}{(q)_{m_1}(q)_{m_1+2n_1-1}} \\ &\quad+ \sum_{n_2=1}^{\infty}\sum_{m_2=0}^{\infty} \frac{q^{m_2^2+m_2(2n_2-1)+\frac{2k(k-\lambda)(2n_2-1)+(k-\lambda)^2+k^2(2n_2-1)^2}{4k}}}{(q)_{m_2}(q)_{m_2+2n_2-1}}\ .\end{aligned} \tag{3.7.18}$$

Setting $n_1 = \frac{m_2-m_1+1}{2}$ and $n_2 = \frac{m_1-m_2+1}{2}$ implies

$$\sum_{\substack{0\le m_1<m_2=0 \\ m_1+m_2\equiv 1 \pmod 2}}^{\infty} \quad \sum_{\substack{0\le m_2<m_1=0 \\ m_1+m_2\equiv 1 \pmod 2}}^{\infty} \frac{q^{\frac{k}{4}(m_1^2+m_2^2)+\frac{2-k}{2}m_1m_2+\frac{k-\lambda}{2}(m_1-m_2)+\frac{(k-\lambda)^2}{4k}}}{(q)_{m_1}(q)_{m_2}}\ . \tag{3.7.19}$$

Thus, from (3.7.17) and (3.7.19),

$$\begin{aligned}\Lambda_{\lambda,k}(\tau) &= \frac{\Theta_{\lambda,k}(\tau)}{\eta(\tau)} \\ &= \sum_{\substack{\vec{m}=0 \\ m_1+m_2\equiv 0 \pmod 2}}^{\infty} \frac{q^{\frac{1}{4}\vec{m}^t\begin{pmatrix}k & 2-k\\ 2-k & k\end{pmatrix}\vec{m}+\frac{1}{2}\begin{pmatrix}\lambda\\ -\lambda\end{pmatrix}^t\vec{m}+\frac{\lambda^2}{4k}-\frac{1}{24}}}{(q)_{\vec{m}}} && (3.7.20a) \\ &= \sum_{\substack{\vec{m}=0 \\ m_1+m_2\equiv 1 \pmod 2}}^{\infty} \frac{q^{\frac{1}{4}\vec{m}^t\begin{pmatrix}k & 2-k\\ 2-k & k\end{pmatrix}\vec{m}+\frac{1}{2}\begin{pmatrix}-(k-\lambda)\\ k-\lambda\end{pmatrix}^t\vec{m}+\frac{(k-\lambda)^2}{4k}-\frac{1}{24}}}{(q)_{\vec{m}}}\ . && (3.7.20b)\end{aligned}$$

This two-fold $q$-hypergeometric series has been given without explicit proof in [KMM93].[11] These are fermionic expressions for (3.7.7b) to (3.7.7d). We obtained the fermionic expressions of the remaining two characters, which were unknown so far, as follows: Note that $\frac{(\partial\Theta)_{1,2}}{\eta^3(q)} = 1$ and hence

$$\chi^{\pm}_{1,2} = \frac{\Theta_{1,2}}{2\eta} \pm \frac{1}{2}\eta^2\ . \tag{3.7.21}$$

We then use an identity

$$\eta(q) = q^{\frac{1}{24}} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n+1}{2}}}{(q)_n} \tag{3.7.22}$$

by Euler (cf. [Zag06] for a simple proof). This identity may be squared, leading to

$$\eta^2(q) = \tilde{\eta}^2(q,-1) \quad \text{with} \quad \tilde{\eta}^2(q,z) = \sum_{\vec{m}=0}^{\infty} \frac{q^{\frac{1}{2}\vec{m}^t\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\vec{m}+\frac{1}{2}\begin{pmatrix}1\\ 1\end{pmatrix}^t\vec{m}+\frac{1}{12}} z^{m_1+m_2}}{(q)_{\vec{m}}}\ . \tag{3.7.23}$$

[11] Note that (3.7.20) is not unique just as (2.5.1): According to (2.5.3) the vector may be changed in certain ways along with the constant.

It is possible to transform the fermionic expression of $\chi_{1,2}$ which was obtained in (3.7.20) into

$$\sum_{\substack{\vec{m}=0\\ m_1+m_2\equiv 0 \pmod 2}}^{\infty} \frac{q^{\frac{1}{2}\vec{m}^t\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\vec{m}+\frac{1}{2}\left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)^t\vec{m}}}{(q)_{\vec{m}}} = \frac{1}{2}\sum_{\substack{\vec{m}=0\\ m_1+m_2\equiv 0 \pmod 2}}^{\infty} \frac{q^{\frac{1}{2}\vec{m}^t\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\vec{m}+\frac{1}{2}\left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)^t\vec{m}}}{(q)_{\vec{m}}} \tag{3.7.24}$$

$$+\frac{1}{2}\sum_{\substack{\vec{m}=0\\ m_1+m_2\equiv 1 \pmod 2}}^{\infty} \frac{q^{\frac{1}{2}\vec{m}^t\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\vec{m}+\frac{1}{2}\left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)^t\vec{m}}}{(q)_{\vec{m}}}$$

$$=\left(\sum_{m_1=0}^{\infty}\frac{q^{\frac{m_1^2-m_1}{2}}}{(q)_{m_1}}\right)\left(\sum_{\substack{m_2=0\\ m_2\equiv m_1 \pmod 2}}^{\infty}\frac{q^{\frac{m_2^2+m_2}{2}}}{(q)_{m_2}}\right) \tag{3.7.25}$$

$$+\left(\sum_{m_1=0}^{\infty}\frac{q^{\frac{m_1^2-m_1}{2}}}{(q)_{m_1}}\right)\left(\sum_{\substack{m_2=0\\ m_2\equiv m_1+1 \pmod 2}}^{\infty}\frac{q^{\frac{m_2^2+m_2}{2}}}{(q)_{m_2}}\right).$$

By using

$$\sum_{m=0}^{\infty}\frac{q^{\frac{m_2^2+m_2}{2}}}{(q)_m} = \frac{1}{2}\sum_{m=0}^{\infty}\frac{q^{\frac{m_2^2+m_2}{2}}}{(q)_m}, \tag{3.7.26}$$

which holds because

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{q^{\frac{m_2^2+m_2}{2}}}{(q)_m} &= \sum_{m=0}^{\infty}\frac{q^{\frac{m_2^2+m_2}{2}}(1-q^{m+1})}{(q)_{m+1}}\\ &= \sum_{m=0}^{\infty}\frac{q^{\frac{m_2^2+m_2}{2}}}{(q)_{m+1}} - \sum_{m=0}^{\infty}\frac{q^{\frac{m_2^2+m_2}{2}+m+1}}{(q)_{m+1}}\\ &= \sum_{m=1}^{\infty}\frac{q^{\frac{m_2^2-m_2}{2}}}{(q)_m} - \sum_{m=1}^{\infty}\frac{q^{\frac{m_2^2+m_2}{2}}}{(q)_{m+1}},\end{aligned} \tag{3.7.27}$$

$\chi_{1,2}$ may be written as

$$q^{\frac{1}{24}}\chi_{1,2} = \left(\sum_{m_1=0}^{\infty}\frac{q^{\frac{m_1^2+m_1}{2}}}{(q)_{m_1}}\right)\left(\sum_{\substack{m_2=0\\ m_2\equiv m_1 \pmod 2}}^{\infty}\frac{q^{\frac{m_2^2+m_2}{2}}}{(q)_{m_2}}\right)$$

$$+\left(\sum_{m_1=0}^{\infty}\frac{q^{\frac{m_1^2+m_1}{2}}}{(q)_{m_1}}\right)\left(\sum_{\substack{m_2=0\\ m_2\equiv m_1+1 \pmod 2}}^{\infty}\frac{q^{\frac{m_2^2+m_2}{2}}}{(q)_{m_2}}\right)$$

$$=\sum_{\vec{m}=0}^{\infty}\frac{q^{\frac{1}{2}\vec{m}^t\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\vec{m}+\frac{1}{2}\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^t\vec{m}}}{(q)_{\vec{m}}}, \tag{3.7.28}$$

finally leading to

$$
\begin{aligned}
q^{\frac{1}{24}}\chi^{\pm}_{1,2} =& \frac{\Theta_{1,2}}{2\eta} \pm \frac{\eta^2}{2} \\
=& \frac{1}{2}\sum_{\vec{m}=0}^{\infty} \frac{q^{\frac{1}{2}\vec{m}^t\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\vec{m}+\frac{1}{2}\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^t\vec{m}}}{(q)_{\vec{m}}} \pm \frac{1}{2}\sum_{\vec{m}=0}^{\infty} \frac{q^{\frac{1}{2}\vec{m}^t\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\vec{m}+\frac{1}{2}\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^t\vec{m}+\frac{1}{12}}(-1)^{m_1+m_2}}{(q)_{\vec{m}}} \\
=& \sum_{\substack{\vec{m}=0 \\ m_1+m_2\equiv a \pmod 2}}^{\infty} \frac{q^{\frac{1}{2}\vec{m}^t\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\vec{m}+\frac{1}{2}\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^t\vec{m}}}{(q)_{\vec{m}}}
\end{aligned}
\tag{3.7.29}
$$

with $a = 0$ if the plus sign is chosen and $a = 1$ if the minus sign is chosen.

Thus, also the remaining two characters yield expressions which consist of only one fundamental fermionic form.

The following is a list of the fermionic expressions for all five characters of the logarithmic conformal field theory model corresponding to central charge $c_{2,1} = -2$ that we presented in our recent article [FGK07]:

$$\chi^{+}_{1,2} = \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^2 \\ m_1+m_2\equiv 0 \pmod 2}} \frac{q^{\frac{1}{2}\vec{m}^t\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\vec{m}+\frac{1}{2}\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^t\vec{m}+\frac{1}{12}}}{(q)_{\vec{m}}} \tag{3.7.30a}$$

$$\chi_{0,2} = \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^2 \\ m_1+m_2\equiv 0 \pmod 2}} \frac{q^{\frac{1}{2}\vec{m}^t\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\vec{m}-\frac{1}{24}}}{(q)_{\vec{m}}} \tag{3.7.30b}$$

$$\chi_{1,2} = \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^2 \\ m_1+m_2\equiv 0 \pmod 2}} \frac{q^{\frac{1}{2}\vec{m}^t\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\vec{m}+\frac{1}{2}\left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)^t\vec{m}+\frac{1}{12}}}{(q)_{\vec{m}}} \tag{3.7.30c}$$

$$\chi_{2,2} = \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^2 \\ m_1+m_2\equiv 0 \pmod 2}} \frac{q^{\frac{1}{2}\vec{m}^t\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\vec{m}+\left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)^t\vec{m}+\frac{11}{24}}}{(q)_{\vec{m}}} \tag{3.7.30d}$$

$$\chi^{-}_{1,2} = \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^2 \\ m_1+m_2\equiv 1 \pmod 2}} \frac{q^{\frac{1}{2}\vec{m}^t\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\vec{m}+\frac{1}{2}\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^t\vec{m}+\frac{1}{12}}}{(q)_{\vec{m}}} \tag{3.7.30e}$$

and also

$$\chi_{1,2} = \sum_{\vec{m}\in(\mathbb{Z}_{\geq 0})^2} \frac{q^{\frac{1}{2}\vec{m}^t\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\vec{m}+\frac{1}{2}\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^t\vec{m}}}{(q)_{\vec{m}}}\,. \tag{3.7.31}$$

Using the equality to the bosonic representation of the characters, these give *bosonic-fermionic q-series identities* generalizing the left and right hand sides of (3.2.5). In (3.7.30b) to (3.7.30d), also the last line of (3.7.20) may be used, where $m_1 + m_2 \equiv 1 \pmod 2$.

It is remarkable that, although two of the characters have inhomogeneous modular weight, there is a uniform representation for all five characters with the same matrix $A$ in every

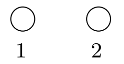



**Figure 3.2.:** The Dynkin diagram of $D_2$

case. But on the other hand, this is a satisfying result, since this is also the case for all other models for which fermionic character expressions are known: Their different modules are only distinguished by the linear term in the exponent, not by the quadratic one. Note that the fact that the quadratic form is diagonal goes well with the description of the $c = -2$ model in terms of symplectic fermions [Kau95, Kau00], see section 5.4.1.

The results are also in agreement with Nahm's conjecture (see section 3.3), which predicts that for a matrix of the form $A = \left(\begin{smallmatrix}\alpha & 1-\alpha\\ 1-\alpha & \alpha\end{smallmatrix}\right)$ with rational coefficients, there exist a vector $\vec{b} \in \mathbb{Q}^r$ and a constant $c \in \mathbb{Q}$ such that $f_{A,\vec{b},c}(\tau) = \sum_{\vec{m}\in(\mathbb{Z}_{\geq 0})^r}^{\infty} \frac{q^{\frac{1}{2}\vec{m}^t A\vec{m}+\vec{b}^t\vec{m}+c}}{(q)_{\vec{m}}}$ is a modular function.

### 3.7.3. Fermionic Character Expressions for $\mathcal{W}(2, 2p-1, 2p-1, 2p-1)$

The matrix $\frac{1}{2}\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ was found in the quadratic form of the fermionic expressions for the $\mathcal{W}(2, 3, 3, 3)$ model at $c = -2$ in the previous section. A generalization to $\mathcal{W}(2, 2p-1, 2p-1, 2p-1)$ is possible by recognizing that the matrix in the case of $p = 2$ is just the inverse of the Cartan matrix of the degenerate case $D_2 = so(4) = A_1 \times A_1$ of the $D_n = so(2n)$ series of simple Lie algebras, where the corresponding Dynkin diagram consists just of two disconnected nodes, as shown in figure 3.7.3. Consequently, one may try the inverse Cartan matrices

$$C_{D_p^{-1}} = \begin{pmatrix} 1 & 1 & \cdots & 1 & \frac{1}{2} & \frac{1}{2}\\ 1 & 2 & \cdots & 2 & 1 & 1\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 1 & 2 & \cdots & p-2 & \frac{p-2}{2} & \frac{p-2}{2}\\ \frac{1}{2} & 1 & \cdots & \frac{p-2}{2} & \frac{p}{4} & \frac{p-2}{4}\\ \frac{1}{2} & 1 & \cdots & \frac{p-2}{2} & \frac{p-2}{4} & \frac{p}{4} \end{pmatrix} \tag{3.7.32}$$

of $D_p = so(2p)$, $p > 2$, for the fermionic expressions of the characters of the $c_{p,1}$ models in the case of $p > 2$. The first thing we noticed by comparing expansions when we tried these matrices in (3.3.1) is that $\vec{b} = 0$ leads to a fermionic expression for $\frac{\Theta_{0,p}}{\eta(q)}$. However, the restriction $m_1 + m_2 \equiv 0 \pmod 2$ has to be changed to $m_{p-1} + m_p \equiv 0 \pmod 2$ implying that particles of the two species corresponding to the two nodes labeled by $n-1$ and $n$ in the $D_n$ dynkin diagram (see figure 3.7.3), which are both connected to the node labeled by $n-2$, may only be created in pairs, as will be shown in detail in chapter 5. These expressions coincide with the ones found in [KKMM93a] (but only the ones with $\vec{b} = 0$), since the characters of the free boson with central charge $c = 1$ and compactification radius $r = \sqrt{\frac{p}{2}}$ [Gin88] equal some of the characters of the $c_{p,1}$ models. We now used our experience on fermionic character expressions to guess the vectors for the rest of the characters of the

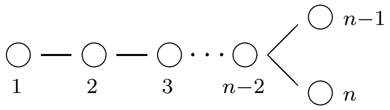



**Figure 3.3.:** The Dynkin diagram of $D_n$

other representations for each given $c_{p,1}$ model. The expressions for $\frac{\Theta_{\lambda,p}}{\eta(q)}$ have $+\frac{\lambda}{2}$ and $-\frac{\lambda}{2}$ in the last two entries of $\vec{b}$ and zero in the other components as is the case for the other, strictly two-dimensional fermionic expression for $\frac{\Theta_{\lambda,p}}{\eta(q)}$ given earlier in (3.7.20).

Still missing now are fermionic expressions for those characters whose bosonic form is of inhomogeneous modular weight, i.e. which consist of theta and affine theta functions. For the vacuum character of $c = -2$ the vector is $\vec{b} = \frac{1}{2}\binom{1}{1}$. Based on our experience with fermionic expressions for other models, we guessed that the vector for the inhomogeneous characters of any $c_{p,1}$ model will have $+\frac{\lambda}{2}$ in both its last components and the rest of the $k-2$ components will increase in integer steps from top to bottom, starting with zero at the component number $i$. The value of $i$ depends on the values of $\lambda$ and $k$. All components above the component number $i$ are zero, too. The detailed description of this vector in dependence of $\lambda$ and $k$ is given below. In this way, we found the fermionic expressions for all characters of all $c_{p,1}$ models and thus a whole new, infinite set of bosonic-fermionic $q$-series identities, given below. In section 5.4.1 and 5.4.2, we will propose a physical interpretation in terms of quasi-particles. Expanding the new fermionic character expressions in $q$, one may convince himself that all coefficients match those of the bosonic character expressions. We checked it up to $k = 5$ and high order.

In short, the fermionic sum representations for all characters of the $\mathcal{W}(2, 2p-1, 2p-1, 2p-1)$, $p \geq 2$, series of triplet algebras corresponding to central charge $c_{p,1}$[12] can be expressed as follows and indeed equal the bosonic ones (cf. (3.7.6a)-(3.7.6f)), the latter being redisplayed on the right hand side for convenience:

$$\chi_{\lambda,k} = \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^k \\ m_{k-1}+m_k\equiv 0 \pmod 2}} \frac{q^{\vec{m}^t C_{D_k}^{-1}\vec{m} + \vec{b}^t_{\lambda,k}\vec{m} + c^\star_{\lambda,k}}}{(q)_{\vec{m}}} = \frac{\Theta_{\lambda,k}}{\eta} \tag{3.7.33a}$$

$$\chi^+_{\lambda',k} = \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^k \\ m_{k-1}+m_k\equiv 0 \pmod 2}} \frac{q^{\vec{m}^t C_{D_k}^{-1}\vec{m} + \vec{b'}^{+t}_{\lambda',k}\vec{m} + c^\star_{\lambda',k}}}{(q)_{\vec{m}}} = \frac{(k-\lambda')\Theta_{\lambda',k} + (\partial\Theta)_{\lambda',k}}{k\eta} \tag{3.7.33b}$$

$$\chi^-_{\lambda',k} = \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^k \\ m_{k-1}+m_k\equiv 1 \pmod 2}} \frac{q^{\vec{m}^t C_{D_k}^{-1}\vec{m} + \vec{b'}^{-t}_{\lambda',k}\vec{m} + c^\star_{k-\lambda',k}}}{(q)_{\vec{m}}} = \frac{\lambda'\Theta_{\lambda',k} - (\partial\Theta)_{\lambda',k}}{k\eta} \tag{3.7.33c}$$

for $0 \leq \lambda \leq k$ and $0 < \lambda' < k$, where $k = p$ since $p' = 1$ and $(\vec{b}_{\lambda,k})_i = \frac{\lambda}{2}(\pm\delta_{i,k-1} \mp \delta_{i,k})$ for $1 \leq i \leq k$, $(\vec{b'}^+_{\lambda',k})_i = \max\{0, \lambda' - (k-i-1)\}$ for $1 \leq i < k-1$ and $(\vec{b'}^+_{\lambda',k})_i = \frac{\lambda'}{2}$ for

[12] This means the characters and not the torus vacuum amplitudes (3.7.6e) and (3.7.6f). Note that $\lim_{\alpha\to 0} \tilde{\chi}^+_{\lambda,k} = \lim_{\alpha\to 0} \tilde{\chi}^-_{\lambda,k} = \chi_{\lambda,k}$ for $0 < \lambda < k$.

$k-1 \leq i \leq k$, $(\vec{b'}^{-}_{\lambda',k})_i = (\vec{b'}^{+}_{k-\lambda',k})_i$ and $c^{\star}_{\lambda,k} = \frac{\lambda^2}{4k} - \frac{1}{24}$.[13] Thus, as in the previous section, the $p \times p$ matrix $A = C^{-1}_{D_p}$ is the same for all characters corresponding to a fixed $p$, i.e. for a fixed model. This is in agreement with previous results on fermionic expressions, since it is known to also be the case for the characters of a given minimal model (see e.g. [Wel05]).

For example, the fermionic expression of the vacuum character of the theory corresponding to central charge $c_{5,1} = -18,2$ would be

$$\chi^{+}_{4,5} = \frac{\Theta_{4,5} + (\partial\Theta)_{4,5}}{5\eta} = \sum_{\substack{\vec{m} \in (\mathbb{Z}_{\geq 0})^5 \\ m_4 + m_5 \equiv 0 \pmod 2}} \frac{q^{\vec{m}^t \begin{pmatrix} 1 & 1 & 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 2 & 2 & 1 & 1 \\ 1 & 2 & 3 & \frac{3}{2} & \frac{3}{2} \\ \frac{1}{2} & 1 & \frac{3}{2} & \frac{5}{4} & \frac{3}{4} \\ \frac{1}{2} & 1 & \frac{3}{2} & \frac{3}{4} & \frac{5}{4} \end{pmatrix} \vec{m} + \begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 2 \end{pmatrix}^t \vec{m} + \frac{91}{120}}}{(q)_{\vec{m}}} \,. \tag{3.7.34}$$

# 3.8. Characters of $\mathcal{W}(2,3k)$

## 3.8.1. Bosonic Character Expressions

One obtains the $\mathcal{W}$-character by summing up the Virasoro characters of all the degenerate representations corresponding to the primary fields that make up the local system of chiral vertex operators, as discussed in section 1.4.2, i.e. the representations corresponding to those fields where the dimensions $h_{r,r} \forall r \in \mathbb{Z}_{\geq 0}$ of any two primary fields differ by integers. Since the general Virasoro character for degenerate representations corresponding to highest weight $h_{r,r}$ is given by

$$\chi_{r,r} := \chi^{\mathrm{Vir}}_{|h_{r,r},c\rangle}(q) = \frac{q^{\frac{1-c}{24}}}{\eta(\tau)} \left(q^{h_{r,r}} - q^{h_{r,-r}}\right) = \frac{1}{\eta(\tau)} \left(q^{n^2 k} - q^{n^2(k+1)}\right) \,, \tag{3.8.1}$$

one can calculate the $\mathcal{W}$-character of the vacuum representation as

$$\begin{aligned} \chi^{\mathcal{W}}_0(\tau) &= \sum_{r \in \mathbb{Z}_{\geq 0}} \chi_{r,r}(\tau) \\ &= \frac{1}{2\eta(\tau)} \sum_{r \in \mathbb{Z}} \left(q^{n^2 k} - q^{n^2(k+1)}\right) \\ &= \frac{1}{2\eta(\tau)} (\Theta_{0,k}(\tau) - \Theta_{0,k+1}(\tau)) \,. \end{aligned} \tag{3.8.2}$$

By the properties of the Jacobi-Riemann $\Theta$-functions under modular transformations (see 2.5), one can read off that the functions $\frac{\Theta_{\lambda,k}}{\eta}$ span a finite-dimensional representation of the modular group. Hence, $\mathcal{W}(2,3k)$ is a rational conformal field theory. Knowing the

[13] Note that in (3.7.33a), also $m_{k-1} + m_k \equiv 1 \pmod 2$ may be used as restriction, but then the vector and the constant change to $\vec{b}_{k-\lambda,k}$ and $c^{\star}_{k-\lambda,k}$, respectively (cf. (3.7.20)).

vacuum character is sufficient for determining all other characters of the theory as well. The characters of $\mathcal{W}(2,3k)$ with central charge $c = 1 - 24k$ are given by

$$\begin{aligned}
\chi^{\mathcal{W}}_{(-1)^\epsilon\lambda} &= \frac{\Theta_{\lambda,k+\epsilon}}{\eta} && \text{representations to} && h_{\frac{\lambda}{2(k+\epsilon)},\frac{(-1)^\epsilon\lambda}{2(k+\epsilon)}}, w = 0 && \text{(3.8.3a)}\\
\chi^{\mathcal{W}}_{0} &= \frac{1}{2\eta}(\Theta_{0,k} - \Theta_{0,k+1}) && \text{vacuum representation} && h_{1,1} = 0, w = 0 && \text{(3.8.3b)}\\
\chi^{\mathcal{W}}_{k+1} &= \frac{1}{2\eta}(\Theta_{0,k} + \Theta_{0,k+1}) && \text{rep. of lowest energy} && h_{0,0} = -k, w = 0 && \text{(3.8.3c)}\\
\chi^{\mathcal{W}}_{(-1)^\epsilon(k+\epsilon),\pm} &= \frac{\Theta_{k+\epsilon,k+\epsilon}}{\eta} && \text{reps with same energy} && h_{\frac{1}{2},(-1)^\epsilon\frac{1}{2}}, \pm w, w \neq 0 && \text{(3.8.3d)}
\end{aligned}$$

for $\epsilon \in \{0,1\}$ and $1 \leq \lambda < k+\epsilon$ [Flo93]. Since $h_{0,0} < h_{1,1}$, these theories are non-unitary. The computation of the characters for $\mathcal{W}(2,8k)$ is analogous and can also be found in [Flo93].

## 3.8.2. Fermionic Sum Representations

The fermionic character representations for $\mathcal{W}(2,3k)$ and $\mathcal{W}(2,8k)$ have already been given in this thesis, by either (3.7.20) or (3.7.33a):

$$\begin{aligned}
\Lambda_{\lambda,k}(\tau) &= \frac{\Theta_{\lambda,k}(\tau)}{\eta(\tau)}\\
&= \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^2\\ m_1+m_2\equiv 0 \pmod 2}} \frac{q^{\frac{1}{4}\vec{m}^t\begin{pmatrix}k & 2-k\\ 2-k & k\end{pmatrix}\vec{m}+\frac{1}{2}\begin{pmatrix}\lambda\\ -\lambda\end{pmatrix}^t\vec{m}+\frac{\lambda^2}{4k}-\frac{1}{24}}}{(q)_{\vec{m}}}\\
&= \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^2\\ m_1+m_2\equiv 1 \pmod 2}} \frac{q^{\frac{1}{4}\vec{m}^t\begin{pmatrix}k & 2-k\\ 2-k & k\end{pmatrix}\vec{m}+\frac{1}{2}\begin{pmatrix}-(k-\lambda)\\ k-\lambda\end{pmatrix}^t\vec{m}+\frac{(k-\lambda)^2}{4k}-\frac{1}{24}}}{(q)_{\vec{m}}}
\end{aligned} \tag{3.8.4}$$

or

$$\chi_{\lambda,k} = \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^k\\ m_{k-1}+m_k\equiv 0 \pmod 2}} \frac{q^{\vec{m}^tC^{-1}_{D_k}\vec{m}+\vec{b}^t_{\lambda,k}\vec{m}+c}}{(q)_{\vec{m}}}\,. \tag{3.8.5}$$

However, the vacuum character and the character to the lowest energy representation both involve $\Theta$-functions with different moduli, so we can just give a linear combination of fundamental fermionic forms for these. It is not known if there exists a set of fermionic expressions for all characters of $\mathcal{W}(2,3k)$ which all feature the same matrix $A$ in their quadratic forms, as is expected [NRT93] to be the case for rational conformal field theories. This is also the case for the *twisted* bosonic $\mathcal{W}(2,3k)$ theories [Flo93], albeit no linear combination of more than one fundamental fermionic form is necessary in that case, since the characters for a given model involve either $\frac{\Theta_{\lambda,k}}{\eta}$ or $\frac{\Theta_{\lambda,k+1}}{\eta}$.

# 4. Dilogarithm Identities

## 4.1. The Rogers Dilogarithm

Dilogarithm identities are relations of the form

$$\frac{1}{\mathrm{L}(1)} \sum_{i=1}^{N} \mathrm{L}(x_i) = d \tag{4.1.1}$$

with $x_i$ an algebraic, $d$ a rational number, $N$ being the size of the matrix $A$ in the fermionic form, and L being the *Rogers dilogarithm* (see e.g. [Lew58, Lew81]), defined for $0 < x < 1$ by

$$\mathrm{L}(x) = \mathrm{Li}_2(x) + \frac{1}{2} \log(x) \log(1-x) \ . \tag{4.1.2}$$

with

$$\mathrm{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} \tag{4.1.3}$$

being the *polylogarithm function* defined for $0 \leq x \leq 1$. The polylogarithm arises e.g. in the computation of quantum electrodynamics corrections to the electrons gyromagnetic ratio. It admits analytic continuation to the complex plane as a multi-valued analytical function of $x$. The *Euler dilogarithm function* $\mathrm{Li}_2(x)$ (Euler 1768) has the integral representation

$$\mathrm{Li}_2(x) = -\int_0^x \mathrm{d}t \frac{\log(1-t)}{t} \ . \tag{4.1.4}$$

Note that the connection to the usual logarithm function is given by $-\log(1-x) = \mathrm{Li}_1(x)$. The Rogers dilogarithm satisfies $\mathrm{L}(1) = \frac{\pi^2}{6}$ as well as the *five term relation*

$$\mathrm{L}(w) + \mathrm{L}(z) + \mathrm{L}(1-wz) + \mathrm{L}(\frac{1-w}{1-wz}) + \mathrm{L}(\frac{1-z}{1-wz}) = 3\,\mathrm{L}(1) \ , \tag{4.1.5}$$

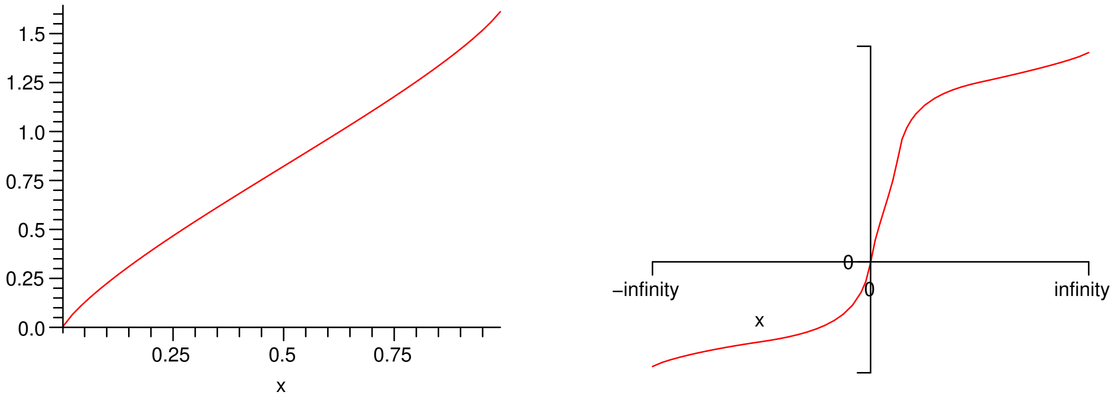
from which it follows that

$$\mathrm{L}(1-z) = \mathrm{L}(1) - \mathrm{L}(z) \ , \tag{4.1.6}$$

$$\mathrm{L}(z^2) = 2\,\mathrm{L}(z) - 2\,\mathrm{L}(\frac{z}{1+z}) \ . \tag{4.1.7}$$

$\mathrm{L}(z)$ may be extended consistently to real $z > 1$ with the help of

$$\mathrm{L}(z) = 2\,\mathrm{L}(1) - \mathrm{L}(\frac{1}{z}) \ . \tag{4.1.8}$$




(a) The Rogers dilogarithm for $0 \leq x \leq 1$

(b) The single-valued Rogers dilogarithm continued to all real numbers

**Figure 4.1.:** The Rogers dilogarithm

Moreover, it is possible to continue the Rogers dilogarithm to $x \in \mathbb{R} \cup \{\pm\infty\}$ by the following rules:

$$\mathrm{L}_{sv}(x) = \begin{cases} \frac{\pi^2}{3} - \mathrm{L}(\frac{1}{x}) & \text{if} \quad x > 1 \\ \mathrm{L}(\frac{1}{1-x}) - \frac{\pi^2}{6} & \text{if} \quad x < 0 \end{cases} , \tag{4.1.9a}$$

$$\mathrm{L}(0) = 0 \,, \quad \mathrm{L}(1) = \frac{\pi^2}{6} \,, \quad \mathrm{L}(+\infty) = \frac{\pi^2}{3} \,, \quad \mathrm{L}(-\infty) = -\frac{\pi^2}{6} \tag{4.1.9b}$$

This is called *single-valued Rogers dilogarithm*, displayed graphically in 4.1(b). Note that this construction does satisfy the relation (4.1.6), but in general not the five term relation (4.1.5), for instance not in the case $x < 0$, $y < 0$ and $xy > 1$. Note also that if a function $f(x)$ is three times differentiable for $0 < x < 1$ and satisfies the five term relation, then it must be the Rogers dilogarithm function (Rogers 1907). It is visualized in figure 4.1(a). The Rogers dilogarithm admits an analytic continuation to the complex plane cut along the real axis from $-\infty$ to 0 and from 1 to $+\infty$.

The dilogarithm and its generalization, the polylogarithm, appear in a lot of branches of mathematics and physics (see e.g. [Kir95]). The Rogers dilogarithm (in the following synonymously just referred to as dilogarithm) is interesting for number theory including algebraic $K$-theory, geometry of hyperbolic 3-manifolds and even Grothendieck's theory of motives. In physics, it occurs for example in the context of integrable 2-dimensional quantum field theories and lattice models: The UV limit or the critical behavior of such systems is typically investigated by methods involving the thermodynamic Bethe ansatz [Bet31, KM90, KM92, KNS93, Zam90]. The central charge of the conformal field theory corresponding to that system can be expressed through the dilogarithm evaluated at certain algebraic numbers.

However, it was demonstrated by e.g. [NRT93, DKK$^+$93, KKMM93a, KKMM93b, Ter92, Ter94] that there is no need to such a physical background of a conformal field theory – dilogarithm identities already arise from the asymptotics of fermionic character expressions using the Richmond-Szekeres method [RS81] and the Kač-Wakimoto theorem. This approach has thenceforward been used to prove some of the various conjectures on dilogarithm identities stated previously in the literature (see [Kir95] for an overview of the identities). In [NRT93], it has been used on the fermionic character expressions for the $c_{2p+1,2}$ minimal Virasoro models ($p \in \mathbb{Z}_{\geq 1}$) In that case, the fermionic expressions are the Andrews-Gordon identities. Applied to the logarithmic conformal field theories corresponding to the $\mathcal{W}(2, 2p-1, 2p-1, 2p-1)$ series of triplet algebras, this approach will in detail be investigated further in this chapter.

## 4.2. Dilogarithm Identities from Saddle-Point Analysis

If a fermionic sum representation for a character and its effective central charge $c_{\text{eff}}$ are known, these will give rise to dilogarithm identities. But furthermore, if one has a $q$-series expressed as a fermionic sum which is not known to correspond to any conformal field theory and thus no effective central charge may be used, it is still possible to extract dilogarithm identities if a product form for that $q$-series is known. The first case is considered here in detail, for the case of the series of triplet algebras $\mathcal{W}(2, 2p-1, 2p-1, 2p-1)$ $\forall p \geq 2$.

Dilogarithm identities for the effective central charges (and conformal dimensions) exist for at least large classes of rational conformal field theories. It is conjectured [NRT93] that all values of the effective central charges occurring in non-trivial rational conformal field theories can be expressed as one of those rational numbers that consist of a sum of an arbitrary number of dilogarithm functions evaluated at algebraic numbers from the interval $(0, 1)$. Thus, the study of dilogarithm identities arising from conformal field theories, e.g. the set of effective central charges that can be expressed with a fixed number $N$ in (4.1.1), gives further insight into the classification of all rational ones.

The place of the constant $d$ in (4.1.1) is then taken by the effective central charge of the conformal field theory. The keynote is that the effective central charge can be determined in different ways: On the one hand, it is fixed by the properties of the character $\chi(q)$ with respect to modular transformations. On the other hand, that number can also be obtained from fermionic character representations by saddle point analysis. Equality of those two expressions results in the often non-trivial dilogarithm identities.

The first way is as follows: The set of character functions of a given rational theory forms a representation of the modular group and under the transformation $\mathcal{S}$, the characters transform [DV88, Kac90] like

$$\chi_j(\tau) = \sum_l S_{j,l}\chi_l(-\frac{1}{\tau}) \ , \tag{4.2.1}$$

implying that

$$\chi_j(\tau) = S_{j,\text{min}} \mathrm{e}^{\frac{\pi c_{\text{eff}}}{12t}} \left[ 1 + \sum_{k \neq \text{min}} \frac{S_{j,k}}{S_{j,\text{min}}} \mathrm{e}^{-\frac{2\pi(h_k - h_{\text{min}})}{t}} + \dots \right] \tag{4.2.2}$$

when $t = -\mathrm{i}\tau \to 0^+$ (i.e. $q \to 1^-$). The value of $c_{\text{eff}}$ in the above relation determines the effective central charge and is given by

$$c_{\text{eff}} = c - 24h_{\text{min}} \ , \tag{4.2.3}$$

where $h_{\text{min}}$ is the smallest of the conformal dimensions of all the fields in the theory.

Here, $S$ represents the modular transformation $\mathcal{S} : \tau \mapsto -\frac{1}{\tau}$ on the space of the character functions, as given in section 2.1. The quotients $\frac{S_{j,i}}{S_{j,\min}} = \mathcal{D}_j^{(i)}$ in the above expression are called *generalized quantum dimensions*.

The second way involves a saddle-point analysis of the fermionic character expressions: Since the leading order of the asymptotic growth of the coefficients $a_M$ in

$$\chi_j(\tau) = q^{h_j - \frac{c}{24}} \sum_{M=0}^{\infty} a_M q^M \tag{4.2.4}$$

is the same for all characters of a given $c_{p,1}$ model, choose (for simplicity) the character $\chi_{0,p}$ of the representation corresponding to $h_{\min}$ with $\vec{b} = 0$

$$\sum_{M=0}^{\infty} a_M q^M = \sum_{\substack{\vec{m} \in \in (\mathbb{Z}_{\geq 0})^p \\ m_{k-1}+m_k \equiv 0 \pmod 2}} \frac{q^{\vec{m}^t B \vec{m}}}{(q)_{\vec{m}}} , \tag{4.2.5}$$

where $\vec{m}^t = (m_1, \ldots, m_p)$ and $B := C_{D_p}^{-1}$ is the inverse of the Cartan matrix of $D_p$ (cf. section 3.7.3).

By Cauchy's theorem, it follows that

$$a_{M-1} = \oint \frac{\mathrm{d}q}{2\pi \mathrm{i}} \sum_{\substack{\vec{m} \in \in (\mathbb{Z}_{\geq 0})^p \\ m_{p-1}+m_p \equiv 0 \pmod 2}} \frac{q^{\vec{m}^t B \vec{m}}}{(q)_{\vec{m}}} , \tag{4.2.6}$$

where the sum restrictions won't affect the result and hence are omitted.

It is possible to roughly approximate the integral through a method called saddle-point approximation, where the integrand is being evaluated at its saddle-point. This is done as follows: At first, replace the summation in the integrand by an integration over $\mathrm{d}m^p$, where the $m_i$ are being treated as continuous variables. In order to find the saddle point, the partial derivatives of the integrand $f(q, \vec{m})$ are being set to zero. Since the logarithm is strictly monotone, one may also use the logarithm of the integrand which will make the evaluation easier:

$$\log f(q, \vec{m}) \simeq (\vec{m}^t B \vec{m} - M) \log q - \sum_{i=1}^{p} \int_0^{m_i} \mathrm{d}p \log(1 - q^p) , \tag{4.2.7}$$

where

$$\log(q)_{m_i} \simeq \int_0^{m_i} \mathrm{d}p \log(1 - q^p) . \tag{4.2.8}$$

Here, use has been made of the Euler-Maclaurin formula, a powerful connection between integrals and sums. Now the saddle point conditions

$$\partial_{m_i} \log f(q, \vec{m}) = (\sum_j B_{ij} m_j + B_{ji} m_j) \log q - \log(1 - q^{m_i}) = 0 \tag{4.2.9}$$

are being applied, where

$$\int \mathrm{d}p \log(1 - q^p) = -\frac{\mathrm{Li}_2(q^p)}{\log q} , \tag{4.2.10}$$

and thus

$$\partial_{m_i} \int_0^{m_i} \mathrm{d}p \log(1-q^p) = \log(1-q^{m_i}) \ . \tag{4.2.11}$$

Exponentiating this expression implies

$$q^{\frac{\sum_j (B_{ij}+B_{ji})m_j}{1-q^{m_i}}} = 1 = \frac{\prod_{j=1}^p q^{(B_{ij}+B_{ji})m_j}}{1-q^{m_i}} \tag{4.2.12}$$

and, consequently,

$$1-q^{m_i} = \prod_{j=1}^p q^{m_j(B_{ij}+B_{ji})} \ . \tag{4.2.13}$$

Or, in more conventional notation,

$$\delta_i = \prod_{j=1}^p (1-\delta_j)^{2B_{ij}} \ , \tag{4.2.14}$$

since $B$ is symmetric. Moreover, $\delta_i = 1-q^{m_i}$. Substituting this into $\log f(q,\vec{m})$ then leads to

$$\log f(q,\vec{m}) = -M \log q - \sum_i \frac{m_i}{\log(1-\delta_i)} \mathrm{L}(\delta_i) \tag{4.2.15}$$

and thus

$$\log f(q,\vec{m}) = -M \log q - \frac{1}{\log q} \sum_{i=1}^p \mathrm{L}(\delta_i) \ , \tag{4.2.16}$$

since by (4.1.2) and (4.1.6)

$$\int_0^{m_i} \mathrm{d}p \log(1-q^p) = \frac{\frac{\pi^2}{6} - \mathrm{Li}_2(q^{m_i})}{\log q} = \frac{\mathrm{L}(1-q^{m_i}) + \frac{1}{2}\log(1-q^{m_i})\log(q^{m_i})}{\log q} \ . \tag{4.2.17}$$

The partial derivative $\partial_q \log f = 0$ now fixes the saddle point

$$(\log q)^2 = \frac{1}{M} \sum_{i=1}^p \mathrm{L}(\delta_i) \tag{4.2.18}$$

with

$$q = \mathrm{e}^{\sqrt{\frac{1}{M}\sum_i \mathrm{L}(\delta_i)}} \ . \tag{4.2.19}$$

Substituting this result into (4.2.16) and again exponentiating, one may evaluate the integrand at its saddle point and find

$$f(q,\vec{m}) \mid_{\text{saddle point}} = \mathrm{e}^{-2\sqrt{M\sum_i \mathrm{L}(\delta_i)}} \ , \tag{4.2.20}$$

implying that the asymptotic behavior of $a_M$ is given by

$$a_M \simeq= \mathrm{e}^{-2\sqrt{M\sum_i \mathrm{L}(\delta_i)}} \ . \tag{4.2.21}$$

Now, considering the character

$$\chi_j(\tau) = q^{h_j - \frac{c}{24}} \sum_{M=0}^{\infty} a_M q^M \ , \tag{4.2.22}$$

one can approximate its $q$-series as

$$\sum_M a_M q^M \simeq \int \mathrm{d}M a_M q^M = 2\mathrm{e}^{\frac{\sum_i \mathrm{L}(\delta_i)}{2\pi t}} \int_0^\infty \mathrm{d}x \ x\mathrm{e}^{-2\pi t(x - \frac{\sqrt{\sum_i \mathrm{L}(\delta_i)}}{2\pi t})^2} \sim \mathrm{e}^{\frac{\sum_i \mathrm{L}(\delta_i)}{2\pi t}} \ . \tag{4.2.23}$$

Finally, upon comparison of this expression with (4.2.2), it is possible to read off that the effective central charge must be given by

$$c_{\mathrm{eff}} = \frac{\sum_{i=1}^k \mathrm{L}(\delta_i)}{\mathrm{L}(1)} \ . \tag{4.2.24}$$

It is thus possible to extract dilogarithm identities from the set of algebraic equations (4.2.14).

## 4.3. The Identities Corresponding to the Triplet $\mathcal{W}$-Algebras

To support the fermionic character expressions we derived in section 3.7.3, we show in this section that it is possible to correctly extract dilogarithm identities from them. The effective central charge of the given logarithmic $\mathcal{W}(2, 2p-1, 2p-1, 2p-1)$ model should be expressible as a sum of dilogarithm functions evaluated at certain algebraic numbers, where these numbers are determined by the matrix $A$ in the quadratic form in the exponent of the fermionic character expression.

The effective central charge of the logarithmic conformal field theories corresponding to central charge $c_{p,1}$ ($p \geq 2$) is given by

$$c_{\mathrm{eff}}^{p,1} = c_{p,1} - 24h_{\mathrm{min}}^{p,1} = 1 \ . \tag{4.3.1}$$

It is remarkable that, although those $c_{p,1}$ theories are non-minimal models on the edge of the conformal grid, it is still possible (numerically solving (4.2.14)) to correctly extract the well-known infinite set of dilogarithm identities

$$2L(\frac{1}{p}) + \sum_{j=2}^{p-1} L(\frac{1}{j^2}) = L(1) \quad \forall \ p \geq 2 \ . \tag{4.3.2}$$

This set of identities can also be found in e.g. [KM90, Kir92] and references therein and can be proved using the five term relation which means that it is *accessible* a fact that supports the fermionic sum representations presented in section 3.7.3 for the characters of the $\mathcal{W}(2, 2p-1, 2p-1, 2p-1)$ triplet algebras.

Note that when $p = 2$, this reduces to (4.1.6).

## 4.4. Other Dilogarithm Identities

The calculation done in the previous section is based on the $c_{p,1}$ series of logarithmic conformal field theories. But it is possible to do a similar derivation of dilogarithm identities with the same method for any conformal field character if a fermionic sum representation for that character is known. Examples of other dilogarithm identities which have been obtained in this manner can be found in e.g. [NRT93, Byt99b]. Some of them are obtained from fermionic expressions for linear combinations of Virasoro characters and are related to the so-called *secondary effective central charge*. An extensive list of dilogarithm identities is given by Anatol Kirillov in [Kir95].

# 5. Quasi-Particle Interpretation

Non-unique bases of the Hilbert spaces in two-dimensional conformal field theories establish the existence of several alternative character formulae.

The original formula, the so-called bosonic representation (cf. section 3.2), which traces back to Feigin and Fuks [FF83] and Rocha-Caridi [RC84], is directly based upon the structure of null vectors, i.e. the invariant ideal is divided out. The occurrence of a factor $(q)_\infty$ in the denominator arises naturally in the construction of Fock spaces using bosonic generators. Indeed, the character of a free chiral boson is given by

$$\chi_B = \sum_{n=0}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} (1-q^n)^{-1} = \frac{1}{(q)_\infty} , \tag{5.0.1}$$

where $p(n)$ is the number of additive partitions of the integer $n$ into integer parts greater than zero which don't have to be distinct. Encoded by the numerator, these spaces are then truncated in a particular way in the general bosonic character expression. The interpretation as partition functions requires these expressions to be modular covariant, which is easily checked when expressing the characters in terms of $\Theta$-functions (cf. section 2.5).

In contrast, the fermionic representations possess a remarkable interpretation in terms of quasi-particles for the states, obeying Pauli's exclusion principle. The character of a free chiral fermion with periodic or anti-periodic boundary conditions is given respectively by

$$\chi_{F,P} = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}m^2 - \frac{1}{2}m}}{(q)_m} \quad \text{or} \tag{5.0.2a}$$

$$\chi_{F,A} = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}m^2}}{(q)_m} . \tag{5.0.2b}$$

In the following section, it will be discussed how this comes about.

The bosonic representations are in general unique, since a natural level gradation in terms of the eigenvalue of $L_0$ is induced by the operators $L_n$. Although the fermionic ones are also obviously graded by their $L_0$ eigenvalue, there is in general more than one fermionic expression for the same character, since different types of generalized exclusion statistics may be imposed which might force different quasi-particle systems to lead to the same fermionic character expression.

## 5.1. Quasi-Particle Representation of Fundamental Fermionic Forms

The general fermionic character expression is a linear combination of fundamental fermionic forms. The characters of various series of rational CFTs, including the $c_{p,1}$ series, can be represented as a single fundamental fermionic form [Wel05, BMS98, DKMM94]. For simplicity, we won't deal with the most general case here, but with a certain specialization.[1]

Fermionic sum representations for characters admit an interpretation in terms of fermionic quasi-particles, as shown in [KM93] (see also [KKMM93a]). This can be easily seen from the fundamental fermionic form[2]

$$\chi(q) = \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^r \\ \text{restrictions}}} q^{\vec{m}^t A \vec{m} + \vec{b}^t \vec{m}} \prod_{a=1}^{r} \begin{bmatrix} ((\mathbb{1}-2A)\vec{m}+\vec{u})_a \\ m_a \end{bmatrix}_q , \tag{5.1.1}$$

with the help of combinatorics: The number of additive partitions $P_M(N, N')$ of a positive integer $N$ into $M$ distinct non-negative integers which are smaller than or equal to $N'$ is stated by [Sta72, p. 23]

$$\sum_{N=0}^{\infty} P_M(N, N') q^N = q^{\frac{1}{2}M(M-1)} \begin{bmatrix} N'+1 \\ M \end{bmatrix}_q , \tag{5.1.2}$$

which in the limit $N' \to \infty$ takes the form

$$\lim_{N'\to\infty} \sum_{N=0}^{\infty} P_M(N, N') q^N = q^{\frac{1}{2}M(M-1)} \frac{1}{(q)_M} . \tag{5.1.3}$$

This formula is tailored to our needs, because the requirement of distinctiveness expresses the fermionic nature of the quasi-particles, i.e. Pauli's exclusion principle. To make use of (5.1.3), (5.1.1) can be reformulated to

$$\chi(q) = \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^r \\ \text{restrictions}}}^{\infty} q^{\frac{1}{2}\sum_{i=1}^{r}(m_i^2 - m_i) + \sum_{i=1}^{r}(b_i+\frac{1}{2})m_i + \sum_{i,j=1}^{r} A_{ij} m_i m_j - \frac{1}{2}\sum_{i=1}^{r} m_i^2} \prod_{a=1}^{r} \begin{bmatrix} ((\mathbb{1}-2A)\vec{m}+\vec{u})_a \\ m_a \end{bmatrix}_q \tag{5.1.4}$$

$$= \prod_{i=1}^{r} \left( \sum_{\substack{m_i \\ \text{restrictions}}}^{\infty} q^{\frac{1}{2}\sum_{i=1}^{r}(m_i^2 - m_i) + (b_i+\frac{1}{2})m_i + \sum_{j=1}^{r} A_{ij} m_i m_j - \frac{1}{2} m_i^2} \right) \prod_{a=1}^{r} \begin{bmatrix} ((\mathbb{1}-2A)\vec{m}+\vec{u})_a \\ m_a \end{bmatrix}_q . \tag{5.1.5}$$

Applying (5.1.3) to the fundamental fermionic form (5.1.1) leads to

$$\prod_{i=1}^{r} \left( \sum_{\substack{m_i \\ \text{restrictions}}}^{\infty} \sum_{N=0}^{\infty} P_{m_i}(N, ((\mathbb{1}-2A)\vec{m}+\vec{u})_a - 1) q^{N + (b_i+\frac{1}{2})m_i + \sum_{j=1}^{r} A_{ij} m_i m_j - \frac{1}{2} m_i^2} \right) . \tag{5.1.6}$$

We can then make use of the relation

$$\sum_{N=0}^{\infty} P_M^0(N, N') q^{N+kM} = \sum_{N=0}^{\infty} P_M^k(N, N'+k) q^N , \tag{5.1.7}$$

[1] This specialization is also called fundamental fermionic form in [BMS98].

[2] A possible constant $c$ has been omitted, since it would just result in an overall shift of the energy spectrum of the resulting quasi-particles.

where we defined $P_M^k(N,N')$ like $P_M(N,N')$ but with the additional requirement that all the integers that make up a partition have to be greater than or equal to $k$. This relation is obvious since it is a one-to-one mapping of partitions and thus nothing more than just a mere shift of the partitions: Each part of a given partition of $N$ into $M$ distinct parts is increased by $k$, which turns $N$ into $N+Mk$. (5.1.7) allows us to rewrite (5.1.6) into

$$\prod_{i=1}^{r}\left(\sum_{\substack{m_i\\ \text{restrictions}}}^{\infty}\sum_{N=0}^{\infty}P_{m_i}^{b_i+\frac{1}{2}+\left((A-\frac{1}{2}\mathbb{1})\vec{m}\right)_i}(N,-(A-\frac{1}{2}\mathbb{1})\vec{m})_a+\vec{b}_a-\frac{1}{2}+\vec{u}_a)q^N\right) \ . \tag{5.1.8}$$

For the quasi-particle interpretation, the characters are regarded as partition functions $Z$ for left-moving excitations with the ground-state energy scaled out

$$\chi\sim Z=\sum_{\text{states}}\mathrm{e}^{-\frac{E_{\text{states}}}{kT}}=\sum_{l=0}^{\infty}P(E_l)\mathrm{e}^{-\frac{E_l}{kT}} \tag{5.1.9}$$

with $T$ being the temperature, $k$ the Boltzmann's constant, $E_l$ the energy and $P(E_l)$ the degeneracy of the particular energy level $l$.

The energy spectrum consists of all the excited state energies (minus the groundstate energy) that are given by

$$E_l=E_{ex}-E_{GS}=\sum_{i=1}^{r}\sum_{\substack{\alpha=1\\ \text{restrictions}}}^{m_i}e_i(p_\alpha^i)\ , \tag{5.1.10}$$

and the corresponding momenta of the states are given by

$$P_{ex}=\sum_{i=1}^{r}\sum_{\substack{\alpha=1\\ \text{restrictions}}}^{m_i}p_\alpha^i\ , \tag{5.1.11}$$

where $r$ denotes the number of different species of particles, $m_i$ the number of particles of species $i$ in the state, $e_i(p_\alpha^i)$ the single-particle energy of the quasi-particle particle $\alpha$ of species $i$ and the subscript 'restrictions' indicates possible rules under which the excitations may be combined. (5.1.10) is referred to as a quasi-particle spectrum in statistical mechanics (naja: see e.g. [McC94]). Quasi means in this context that for example magnons or phonons have other properties than real particles like protons or electrons. And in addition, the spectrum above may contain single-particle energy levels that are different from the form in relativistic quantum field theory $e_\alpha(p)=\sqrt{M_\alpha^2+p^2}$. This means that if we assume massless single-particle energies

$$e_i(p_\alpha^i)=e(p_\alpha^i)=vp_\alpha^i \tag{5.1.12}$$

($v$ referred to as the fermi velocity, spin-wave velocity, speed of sound or speed of light), where $p_\alpha^i$ denotes the quasi-particle $\alpha$ of 'species' $i$ ($1\leq i\leq r$), and if in (5.1.8) we set

$$q=\mathrm{e}^{-\frac{v}{kT}}\ , \tag{5.1.13}$$

we can read off that the partition function corresponds to a system of quasi-particles that are of $r$ different species and which obey the Pauli exclusion principle

$$p_\alpha^i\neq p_\beta^i\quad\text{for }\alpha\neq\beta\quad\text{and all}\quad i\ , \tag{5.1.14}$$

in order to satisfy Fermi statistics, but whose momenta $p_\alpha^i$ are otherwise freely chosen from the sets

$$P_i = \{p^i_{\min}, p^i_{\min}+1, p^i_{\min}+2, \ldots, p^i_{\max}\} \tag{5.1.15}$$

with minimum momenta

$$p^i_{\min}(\vec{m}) = \left[((A-\frac{1}{2})\vec{m})_i + b_i + \frac{1}{2}\right] \tag{5.1.16}$$

and with the maximum momenta

$$p^i_{\max}(\vec{m}) = -((A-\frac{1}{2}\mathbb{1})\vec{m})_i + (\vec{b})_i - \frac{1}{2} + (\vec{u})_i = -p^i_{\min}(\vec{m}) + 2(\vec{b})_i + (\vec{u})_i \ . \tag{5.1.17}$$

Thus, $p^i_{\max}$ either infinite if $(\vec{u})_i$ is infinite or finite and dependent on $\vec{m}$, $A$, $(\vec{b})_i$ and $(\vec{u})_i$. Note that if $(\vec{u})_i$ is infinite for all $i \in \{1, \ldots, r\}$, then (5.1.1) reduces to the form (3.3.1) of Nahm's conjecture. Of course, (5.1.1) is only an often encountered specialization of the most general fundamental fermionic form, since the components of the $q$-binomial coefficient may be of a different shape than that given in (5.1.1), but the generalization of the previous steps is obvious. To sum up, this means that a multi-particle state with energy $E_l$ may consist of exactly those combinations of quasi-particles of arbitrary species $i$, whose single-particle energies $e(p^i)$ add up to $E_l$ and where Pauli's principle holds for any two quasi-particles of that combination which belong to the same species. Possible sum restrictions then result in the requirement that certain particles may only be created in conjunction with certain others. Thus, the characters (5.0.2a) and (5.0.2b) of the free chiral fermion with respectively periodic or anti-periodic boundary conditions are obtained in the case of $r=1$, $p_{\max}=\infty$, $p^{(P)}_{\min}=0$ and $p^{(A)}_{\min}=\frac{1}{2}$. On the other hand, the character (5.0.1) of a free chiral boson is obtained by setting $r=1$, $p_{\min}=1$ and $p_{\max}=\infty$ and simply not imposing any exclusion rules, i.e. not using (5.1.2).

Although the upper momentum boundaries may seem artificial, the phenomenon that the momenta $p^\alpha_{i_\alpha}$ for $2 \le \alpha \le n$ are restricted to take only a finite number of values for given $\vec{m}$ is a common occurence in quantum spin chains.

## 5.2. Quasi-Particle Interpretation of Unitary Minimal Models

As we discussed in section 3.4.3, the fermionic expressions for the vacuum characters of the unitary series of minimal models $\mathcal{M}(p,p+1)$, $p \ge 2$, can in general be determined to be [BMS98, Wel05]

$$q^\alpha \chi^{p+1,p}_{1,1} = \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\ge 0})^{p-2} \\ m_i \text{ even}}} \frac{q^{\frac{1}{4}\vec{m}^t C_{A_{p-2}}\vec{m}}}{(q)_{m_1}} \prod_{i=2}^{p-2} \begin{bmatrix} ((\mathbb{1}-\frac{1}{2}C_{A_{p-2}})\vec{m})_i \\ m_i \end{bmatrix}_q \tag{5.2.1}$$

with $\alpha \in \mathbb{Q}$ some constant depending on the parameters of $\chi$. Thus, the unitary minimal model $\mathcal{M}(p,p+1)$ can be represented by a system of $p-2$ quasi-particle species which have the minimum momenta

$$p^i_{\min}(\vec{m}) = \frac{1}{4}((C_{A_{p-2}} - 2\mathbb{1})\vec{m})_i + \frac{1}{2} \ . \tag{5.2.2}$$

Here, the first species is dominant, its spectrum being unbounded from above. The momenta of the other quasi-particle species are restricted to have maximum momenta

$$p^i_{\max}(\vec{m}) = -p^i_{\min}(\vec{m}) \quad \text{for } 2 \leq i \leq p-2 \; . \tag{5.2.3}$$

## 5.3. The Different Quasi-Particle Interpretations of the Ising Model

As shown in section 1.3.3, the Ising model corresponds to the minimal model $\mathcal{M}(4,3)$. When comparing the fermionic expression

$$q^{\frac{1}{48}} \chi^{4,3}_{1,1} = \sum_{\substack{m=0 \\ m \text{ even}}}^{\infty} \frac{q^{\frac{1}{2}m^2}}{(q)_m} \tag{5.3.1}$$

from the previous section (see also (3.4.17)) with the fermionic expression

$$q^{\frac{1}{48}} \chi^{4,3}_{1,1} = \sum_{\vec{m} \in (\mathbb{Z}_{\geq 0})^8} \frac{q^{\vec{m}^t C^{-1}_{E_8} \vec{m}}}{(q)_{\vec{m}}} \tag{5.3.2}$$

from section 3.5, we find two different systems of quasi-particles which both realize the Ising model. The first system ($A_1$) consists of just one quasi-particle species whose specimen may only be created in pairs and the second system ($E_8$) consists of eight quasi-particle species whose members may combined freely – nevertheless obeying Pauli's exclusion principle in both cases, of course. For the $A_1$ system, the minimum momentum is given simply by

$$p_{\min} = \frac{1}{2} \; , \tag{5.3.3}$$

whereas the minimum momenta for the $E_8$ system are given by

$$p^i_{\min} = ((C^{-1}_{E_8} - \frac{1}{2}\mathbb{1})\vec{m})_i + \frac{1}{2} \; , \tag{5.3.4}$$

i.e. $p^8_{\min} = 6m_1 + 10m_2 + 12m_3 + 15m_4 + 18m_5 + 20m_6 + 24m_7 + \frac{59}{2}m_8 + \frac{1}{2}$.

## 5.4. Quasi-Particle Interpretation of the Triplet $\mathcal{W}$-Algebras

In section 3.7.3, we reported about the fermionic character expressions, which we presented in [FGK07], for the series of triplet $\mathcal{W}$-algebras. In this section, we discuss the quasi-particle content, which one can derive from the fermionic expressions, of the $c_{p,1}$ logarithmic conformal field theories, which have $\mathcal{W}(2, 2p-1, 2p-1, 2p-1)$ as symmetry algebras.

### 5.4.1. The $c = -2$ Model

We start with the case $p = 2$, i.e. $c_{2,1} = -2$. In contrast to the characters for e.g. the minimal models, these characters are the traces over the representation modules of the triplet $\mathcal{W}$-algebra, instead of the Virasoro algebra only. However, although highest weight states are labeled by two highest weights in this case, $h$ and $w$ as the eigenvalues of $L_0$ and $W_0$ respectively, we consider only the traces of the operator $q^{L_0-\frac{c}{24}}$. It turns out that these $\mathcal{W}$-characters are given as infinite sums of Virasoro characters, for example [Flo96]

$$\chi_{|0\rangle} = \sum_{k=0}^{\infty}(2k+1)\chi^{\mathrm{Vir}}_{|h_{2k+1,1}\rangle} \ . \tag{5.4.1}$$

Let us now come to the vacuum character (3.7.30a) for the $c_{2,1}$ model, which features the interesting sum restriction $m_1 + m_2 \equiv 0 \pmod 2$ expressing the fact that particles of type 1 and 2 must be created in pairs. Thus, the existence of one-particle states for either particle species is prohibited. Therefore, the single-particle energies must be extracted out of the observed multi-particle energy levels.

Applying (5.1.3) to the fermionic sum representation (cf. also (3.7.30a))

$$q^{-\frac{1}{12}}\chi^{+}_{1,2} = \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^2 \\ m_1+m_2\equiv 0 \pmod 2}} \frac{q^{\frac{1}{2}\vec{m}^t\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)\vec{m}+\frac{1}{2}\left(\begin{smallmatrix}1\\ 1\end{smallmatrix}\right)^t\vec{m}}}{(q)_{\vec{m}}} \tag{5.4.2}$$

of the vacuum character leads to

$$\chi^{+}_{1,2} = \left(\sum_{m_1=0}^{\infty}\sum_{N=0}^{\infty}P_{m_1}(N)q^{N+m_1}\right)\left(\sum_{\substack{m_2=0 \\ m_2\equiv m_1 \pmod 2}}^{\infty}\ \sum_{N=0}^{\infty}P_{m_2}(N)q^{N+m_2}\right) , \tag{5.4.3}$$

where the constant $c$ has been omitted, since it would just result in an overall shift of the energy spectra. Using massless single-particle energies (5.1.12) and setting (5.1.13) in (5.1.8) then results in the partition function (5.1.9) corresponding to a system of two quasi-particle species, with both species having the momentum spectrum $\mathbb{N}_{\geq 1}$, i.e. a multi-particle state with energy $E_l$ may consist of exactly those combinations of an even number of quasi-particles, having momenta $p^i_\alpha$ ($i \in \{1, 2\}$), whose single-particle energies $e(p^i_\alpha)$ add up to $E_l$ and where the momenta $p^i_\alpha \in \mathbb{N}_{\geq 1}$ of each two of the quasi-particles in that combination are distinct unless they belong to different species, i.e. they respect the exclusion principle. Formally, these spectra belong to two free chiral fermions with periodic boundary conditions. Note in this context the physical interpretations in [Kau95, Kau00], in which the CFT for $c_{2,1} = -2$ is generated from a symplectic fermion, a free two-component fermion field of spin one.

### 5.4.2. The p > 2 Relatives

Besides the best understood LCFT with central charge $c_{2,1} = -2$, we now have a look at the quasi-particle content of its $c_{p,1}$ relatives.

The restrictions $m_{p-1} + m_p \equiv Q \pmod 2$ ($Q$ can be thought of as denoting the total charge of the system) in (3.7.33a) to (3.7.33c) imply that the quasi-particles of species $p - 1$

and $p$ are charged under a $\mathbb{Z}_2$ subgroup of the full symmetry of the $D_p$ Dynkin diagram [KKMM93a], while all the others are neutral. This charge reflects the $\mathfrak{su}(2)$ structure carried by the triplet $\mathcal{W}$-algebra such that all representations must have ground states, which are either $\mathfrak{su}(2)$ singlets or $\mathfrak{su}(2)$ doublets. In comparison to the $c_{2,1} = -2$ model, there exist $p$ quasi-particles in each member of the $c_{p,1}$ series, exactly two of which can only be created in pairs, while the others do not have this restriction. These observations suggest the following conjecture: The $c_{p,1}$ theories might possess a realization in terms of free fermions such that they are generated by one pair of symplectic fermions and $p-2$ ordinary fermions. Realizations of that kind are unknown so far, except for the well-understood case $p = 2$, and are a very interesting direction of future research.

Contrary to the case of $p = 2$, the quasi-particles do not decouple here: The minimal momenta for the quasi-particle species can be read off (5.1.16) and are given by

$$p^i_{\min}(\vec{m}) = \begin{cases} -\frac{1}{2}m_i + \sum_{j=1}^{i} jm_j + \sum_{j=i+1}^{p-2} im_j + \frac{i}{2}(m_{p-1}+m_p) + i + \frac{1}{2} \text{ for } 1 \leq i \leq p-2 \\ -\frac{1}{2}m_i + \sum_{j=1}^{p-2} \frac{j}{2}m_j + \frac{p-1}{2} + \frac{1}{2} + \begin{cases} (\frac{p}{4}m_{p-1} + \frac{2-p}{4}m_p) \text{ for } i = p-1 \\ (\frac{2-p}{4}m_{p-1} + \frac{p}{4}m_p) \text{ for } i = p \end{cases} \end{cases}. \tag{5.4.4}$$

Hence, they depend on the numbers of quasi-particles of the different species in the state. But as in the case of $p = 2$, the momentum spectra are not bounded from above.

## 5.5. Quasi-Particle Interpretation of the $SU(2)$ WZW Model

In [FGK07], we presented the following fermionic sum representation for the character of the $SU(2)$ WZW model at level $k-2$, which we reported about in section 3.6.2:

$$\frac{(\partial\Theta)_{k-1,k}(\tau)}{\eta^3(\tau)} = \sum_{\vec{m}\in(\mathbb{Z}_{\geq 0})^{2(k-2)}} \frac{q^{\vec{m}^t(C_{A_2}\otimes C^{-1}_{T_{k-2}})\vec{m}+(\vec{a}_2\otimes\vec{b}_{k-2})^t\vec{m}+c^\flat_{k-1,k}}}{(q)_{\vec{m}}} \tag{5.5.1}$$

with $\vec{a}^t_2 = (1,-1)$ and $\vec{b}^t_k = (1,2,3,\ldots,k)$, and with integer or half-integer spin $l \leq \frac{k}{2}$, $\lambda = k-1 = 2l+1$, i.e. $l = \frac{k}{2} - 1$. If we apply the quasi-particle interpretation that was introduced in section 5.1 to this character expression, then we obtain a system of $2(k-2)$ different quasi-particle species for which Pauli's exclusion principle holds. The momentum spectrum of each particle species is unbounded from above and the minimum momenta are given by (5.1.16).

# Conclusion and Outlook

The $\mathcal{W}$-character functions of the $c_{2,1} = -2$ logarithmic conformal field theory model are somewhat unusual because indecomposable representations are involved. The aim of this work was to investigate whether there exist so-called fermionic sum representations for these $\mathcal{W}$-characters, which admit a realization of the underlying theory in terms of a system of different fermionic quasi-particle species. Such representations were unknown, but because it was found out that the $c_{p,1}$ theories constitute rational[3] theories with respect to the maximally extended symmetry algebra [GK96, CF06], and since there is a conjecture by Werner Nahm that the characters of all rational conformal field theories have a fermionic sum representation [NRT93], it was hoped for that a uniform fermionic expression for all characters of a given $c_{p,1}$ model does exist.

Indeed, we found a uniform fermionic expression for all characters of the $c_{2,1}$ model and were able to prove the resulting new set of bosonic-fermionic $q$-series identities in section 3.7.2. But we found much more than that.

**Our main achievement** is that we found fermionic expressions for all characters of each $c_{p,1}$ model, admitting an interpretation in terms of $p$ fermionic quasi-particle species and thus providing further evidence for the well-definedness of the logarithmic conformal field theories corresponding to central charge $c_{p,1}$, leading to an infinite set of new bosonic-fermionic $q$-series identities generalizing (3.2.5). These expressions were unknown so far and are given in section 3.7.3. We reported the results in our recent article *Fermionic Expressions for the Characters of* $c_{p,1}$ *Logarithmic Conformal Field Theories* [FGK07], which has been accepted for publication in the journal Nuclear Physics B.

Despite the inhomogeneous structure of the bosonic character expressions in terms of modular forms, there exist fermionic quasi-particle sum representations with the same matrix $A$ (cf. (3.3.1)) for all characters of each $c_{p,1}$ model. In particular, the matrix $A$ turned out to be the inverse of the Cartan matrix of the simply-laced Lie algebra $D_p$, where $p = 2$ can be understood as the degenerate case that is isomorphic to two times the Lie algebra $A_1$. Therefore, those expressions fit well into the known scheme of fermionic character expressions for other (standard) conformal field theories.

As discussed in section 5.1, fermionic character expressions imply a realization of the underlying theory in terms of systems of fermionic quasi-particles. The case of the $\mathcal{W}(2, 2p-1, 2p-1, 2p-1)$ series of triplet algebras has been detailed in sections 5.4.1 and 5.4.2. In the $c_{2,1} = -2$ model, i.e. $p = 2$, the quasi-particle interpretation with the sum restriction $m_1 + m_2 \equiv Q \pmod 2$ ($Q \in \{0, 1\}$) implies that there exist two fermionic quasi-particle species whose members may only be created in pairs, i.e. either a pair of particles from the same species or one particle from each species. This goes well with the realization of the $c = -2$ theory in terms of a pair of symplectic fermions [Kau95, Kau00], which is a free two-component fermion field of spin one. In the general $c_{p,1}$ model, it turns out there

[3] in the generalized sense discussed in section 1.4.3

is a set of $p-2$ fermionic quasi-particle species, the members of which may – aside from Pauli's exclusion principle – be combined freely in building an arbitrary multi-particle state, and additionally a set of two species, the members of which may only be created in pairs, with the total charge $Q$ depending on the sector of the theory. This interpretation suggests that the $c_{p,1}$ theories might possess a realization in terms of free fermions such that they are generated by $p-2$ ordinary fermions and one pair of symplectic fermions. Such realizations are unknown so far, except for the well-understood case $p=2$, and are a very interesting direction of future research.

In all cases except $p=2$, the possible quasi-particle momenta obey non-trivial restrictions (5.1.16) for their minimum momenta, depending on the numbers of quasi-particles of each species in the state. Moreover, since the fermionic character expressions are of the form (3.3.1) for all $p \geq 2$, the momentum spectra are unbounded from above.

Since they correspond to rational conformal field theories, it is furthermore satisfying that the obtained fermionic character expressions for the $\mathcal{W}(2, 2p-1, 2p-1, 2p-1)$ theories – although the latter constitute non-minimal models on the edge of the conformal grid – lead correctly to a well-known infinite set of dilogarithm identities, which strongly supports the obtained fermionic expressions. This derivation is explained in detail in chapter 4.

Aside from that, we found a new fermionic expression for the Kač-Peterson [KP84] characters of the $SU(2)$ Wess-Zumino-Witten model at level $k$ which implies an interpretation in terms of $2(k-2)$ different quasi-particle species obeying non-trivial restrictions for their minimum momenta, as argued in sections 3.6.2 and 5.5.

In all cases, it is a remarkable fact that many matrices $A$ that occur in fermionic character expressions of the type (3.2.6) are related to the $ADE$ series of simple Lie algebras. In many cases, $A$ is simply the inverse Cartan matrix or, more generally, the tensor product of a Cartan matrix and an inverse Cartan matrix. However, the origin of this connection is still not understood, indicating that there is a bigger picture which is waiting to be unravelled.

It is worth mentioning that there are also numerous other avenues towards finding fermionic expressions than the ones used in this thesis, among these are Bailey's lemma [Bai49], thermodynamic Bethe ansatz [KNS93], Kostka Polynomials and Hall-Littlewood Functions [SW99, DLT94, AKS05, War02, Mac79, Mac97].

A specific new technique involves quantum groups, crystal bases and finite paths [HKK$^+$98, HKO$^+$99, HKO$^+$01, Dek06]. The so-called *quantum groups* [Kas90, Kas95, HK02, SKAO96] deepen the understanding of symmetry in systems with an infinite number of degrees of freedom. In general, the connection of quantum groups to conformal field theory is given by the *Kazhdan-Lusztig correspondence* which conjectures the equivalence of integral parts of a conformal field theory and corresponding quantum groups. In particular, the Kazhdan-Lusztig-dual quantum group to the logarithmic $\mathcal{W}$-algebras is identified (see e.g. [FGST05, FGST06c, FGST06a, FGST06b]).

As discussed in section 3.4.3, fermionic character expressions are not unique: In general, there exist more than one fermionic expression of the type (3.2.6) for a given character, sporting different matrices $A$ in their quadratic forms. Different expressions imply different quasi-particle interpretations. For example, as argued in section 3.5, the Ising model is expressible in terms of one quasi-particle species and also in terms of eight quasi-particle species. But the different fermionic expressions have more than just a mathematical relevance, e.g. the different quasi-particle interpretations of the Ising model characters fit well into different known physical interpretations, as discussed in section 3.5. This indicates that different fermionic expressions might point to different integrable massive extensions of conformal field theories.

However, if one tries to give physical meaning to the generalized quasi-particle momentum restrictions, conventional second quantizations fails, as argued by Laughlin in the context of the fractional quantum hall effect [Lau83, TSG82]. As discussed in section 3.6.2, Haldane [Hal91] put up generalized exclusion statistics in terms of spinons which can not be described by second quantization. But these shortcomings are not restricted to condensed matter physics. In particle physics, it is long known that the conventional second quantization in quantum field theory has some major drawbacks. String theory was originally introduced in trying to understand the strong interactions, albeit this later became known as the confining phase of quantum chromodynamics. Thus, the limits of second quantization become more and more apparent. As Barry McCoy wrote in [KMM93], "the fact that mathematicians, high-energy physicists, condensed matter physicists and physicists studying statistical mechanics are all contemplating the same abstract object is a truly remarkable demonstration that the whole is much more than the sum of its parts. The synthesis will be achieved when language can be developed that incorporates all aspects of the phenomena at the same time."

# A. Rogers-Ramanujan and Andrews-Gordon Identities

Prof Hardy, who got the very talented mathematician Srinivasa Ramanujan Aiyangar from India to England, said about the Rogers-Ramanujan identities that they are “as remarkable as any which Ramanujan ever wrote down“. A detailed biography of the remarkable life of the commonly acclaimed mathematical genius has been written by Robert Kanigel [Kan91] and a reprint of fotographs of some of his notes is published in [RA88]. As of this writing, two movies are being planned about Ramanujan.

Rogers, Ramujan and Schur discovered independently the analytical statement of the *Rogers-Ramanujan identities*

$$\frac{1}{\prod\limits_{n>=1}(1-q^{5n-1})(1-q^{5n-4})} = 1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)\dots(1-q^n)} \tag{A.1}$$
$$= 1+q+q^2+q^3+2q^4+2q^5+3q^6+3q^7+4q^8+5q^9+6q^{10}+7q^{11}+9q^{12}$$
$$+10q^{13}+12q^{14}+14q^{15}+17q^{16}+19q^{17}+23q^{18}+26q^{19}+31q^{20}+\dots$$

and

$$\frac{1}{\prod\limits_{n>=1}(1-q^{5n-2})(1-q^{5n-3})} = 1+\sum_{n=1}^{\infty}\frac{q^{n^2+n}}{(1-q)\dots(1-q^n)} \tag{A.2}$$
$$= 1+q^2+q^3+q^4+q^5+2q^6+2q^7+3q^8+3q^9+4q^{10}+4q^{11}+6q^{12}$$
$$+6q^{13}+8q^{14}+9q^{15}+11q^{16}+12q^{17}+15q^{18}+16q^{19}+20q^{20}+\dots\ .$$

In 1894, Leonard James Rogers (1862-1933) also derived a number of other identities of this kind [Rog94]. W.N. Bailey developed new methods to obtain new identities, the so-called Bailey flow [Bai47, Bai49]. In 1952, L.J. Slater gave a comprehensive list of about 130 identities of the Rogers-Ramanujan type [Sla51, Sla52].

On the other hand, a combinatorial statement would be given as follows: Euler found the generating function of (additive) partitions $p(n)$ of an integer $n \geq 0$ into an arbitrary number of summands, each greater than or equal to one (and ordered by their size),

$$\prod_{n=1}^{\infty}(1-q^n)^{-1} = \sum_{n=0}^{\infty}p(n)q^n\ . \tag{A.3}$$

Moreover,

$$\prod_{n\in H}^{\infty}(1-q^H)^{-1} = \sum_{n=0}^{\infty}p_H(n)q^n \tag{A.4}$$

is the generating function for the number of partitions of $n \geq 0$ into summands from the set $H$, where the same summand may also appear more than once. This means that the left hand sides of the Rogers-Ramanujan identities have the following combinatorial interpretation: (A.1) is the generating function for the number of partitions of $n$ into parts from the set $H = \{1, 4, 6, 9, 11, 14, 16, \ldots\}$ and (A.2) is the generating function for the number of partitions of $n$ into parts from $H = \{2, 3, 7, 8, 12, 13, 17, \ldots\}$. The right hand sides have the intepretation: (A.1) generates the number of partitions of an integer $n$ in which the difference between any two parts of a partition is greater than or equal to two, while (A.2) generates the number of partitions of $n$ in which the difference of any two parts of a partition is greater than or equal to two and in which each part is greater than one.

In this aspect, finding more identities of the Rogers-Ramanujan type amounts to finding combinatorial identities of the general form: For all positive integers $n$, the partitions of $n$ into parts from a set of residue classes are equinumerous with the partitions of $n$ into parts subject to some restrictions on the difference between parts. [Bre80]

However, using the method from 5.1, it is also possible to interpret the Rogers-Ramanujan identities in terms of fermionic quasi-particles. The minimum momenta (5.1.16) are $p_{\min} = \frac{m+1}{2}$ for (A.1) and $p_{\min} = \frac{m+3}{2}$ for (A.2). Then, the quasi-particle statement for the right-hand sides of (A.1) and (A.2) would be a system of all multi-particle states of all numbers $m \in \mathbb{Z}_{\geq 0}$ of particles of one species whose momenta add up to $n$ and where a given multi-particle state obeys Pauli's exclusion principle, i.e. any two particles must have different momenta from the set $\mathbb{Z}_{\geq 0} + p_{\min}$.

A generalization of the Rogers-Ramanujan identities is stated by the *Andrews-Gordon identities*

$$\sum_{n_1,\ldots,n_k=0}^{\infty} \frac{q^{N_1^2+\ldots+N_k^2+N_a+\ldots+N_k}}{(q)_{n_1}\cdots(q)_{n_k}} = \prod_{\substack{n\not\equiv 0 \pmod{2k+3} \\ n\not\equiv \pm a \pmod{2k+3}}} (1-q^n)^{-1} \tag{A.5}$$

$$= \frac{1}{(q)_\infty} \sum_{m\in\mathbb{Z}} (-1)^m q^{\frac{1}{2}(m(m+1)(2k+3)-am)} \tag{A.6}$$

by George E. Andrews and Basil Gordon, where $N_i = n_i + \ldots + n_k$, $(q)_n := (q;q)_n$ and $(x;q)_n := \prod_{i=0}^{n-1}(1-q^i x)$. This identity can be interpreted combinatorially as a straightforward generalization in the same way as the Rogers-Ramanujan identities, which can be obtained from the Andrews-Gordon identities by setting $k = 1$ and $a = 1$ or $a = 2$ in (A.5). There are different approaches to the Andrews-Gordon identities. Four different proofs exist: an analytic, an algebraic, a combinatorial and a group-theoretical one. A short analytical proof was given by David Bressoud [Bre80]. Comprehensive introductions to number theory and the theory of partitions can be found in [And84, Har79].

# B. The Jacobi Triple Product Identity

This section provides a proof of the Jacobi triple product identity by comparison of the character of the irreducible vacuum representation of a system of charged free fermions computed from a fermionic basis with the character of the bosonized system computed from a bosonic basis. For more details, the reader is referred to [Kac96].

A system of so-called *charged free fermions* can be described by the formal distributions $\psi^{\pm}(z) = \sum_{n \in \mathbb{Z}+\frac{1}{2}} \psi^{\pm}_n z^{-n-\frac{1}{2}}$ with operator product expansions

$$\psi^{\pm}(z)\psi^{\mp}(w) = \frac{1}{z-w} \quad \text{and} \quad \psi^{\pm}(z)\psi^{\pm}(w) = 0 \tag{B.1}$$

corresponding to commutation relations

$$[\psi^{\pm}_m, \psi^{\mp}_n] = \pm\delta_{m+n+1,0} \quad \text{and} \quad [\psi^{\pm}_m, \psi^{\pm}_n] = 0 \ . \tag{B.2}$$

On the vacuum vector,

$$\psi^{\pm}_n|0\rangle = 0 \quad \forall\ n > 0 \ . \tag{B.3}$$

The system admits a one-parameter family of Virasoro fields (i.e. stress-energy tensors) $L^{\lambda}(z) = \sum_{n\in\mathbb{Z}} L^{\lambda}_n z^{-n-2}$ ($\lambda \in \mathbb{C}$)

$$L^{\lambda}(z) = (1-\lambda)N(\partial\psi^{+}(z)\psi^{-}(z)) + \lambda N(\partial\psi^{-}(z)\psi^{+}(z)) \ , \tag{B.4}$$

which satisfy the commutation relations

$$[L^{\lambda}_0, \psi^{+}_m] = (-m-1+\lambda)\psi^{+}_m \quad \text{and} \tag{B.5}$$
$$[L^{\lambda}_0, \psi^{-}_m] = (-m-\lambda)\psi^{-}_m \ . \tag{B.6}$$

$\psi^{\pm}(z)$ have conformal dimensions $\frac{1}{2}$ with respect to the stress-energy tensor. $N$ means the normal ordering prescription. The free charged fermions system may be *bosonized* by

$$\alpha(z) = N(\psi^{+}\psi^{-}) \ , \tag{B.7}$$

which results in $\alpha(z)$ being a field of conformal weight 1. If $\alpha(z) = \sum_{n\in\mathbb{Z}} \alpha_n z^{n-1}$ then

$$[\alpha_m, \alpha_n] = m\delta_{m+n,0} \tag{B.8}$$
$$[\alpha_m, \psi^{\pm}_n] = \pm\psi^{\pm}_{m+n} \ , \tag{B.9}$$

where $\alpha_0$ is called the *charge operator* and the stress-energy tensor can be expressed in the bosonic field as

$$L^{\lambda}(z) = \frac{1}{2}N(\alpha(z)\alpha(z)) + (\frac{1}{2}-\lambda)\partial\alpha(z) \ . \tag{B.10}$$

The irreducible representation $M$ constructed on $|0\rangle$ by action of the modes of the free charged fermions has the 'fermionic' basis

$$\{\psi^-_{j_t}\cdots\psi^-_{j_1}\psi^+_{i_s}\cdots\psi^+_{i_1}|0\rangle \mid 0 > i_1 > i_2 > \ldots,\ 0 > j_1 > j_2 > \ldots\}\ , \tag{B.11}$$

which consists of eigenvectors to the charge operator $\alpha_0$ with eigenvalues $s-t$, called *charges*. The character

$$\chi_M = \mathrm{Tr}_M(q^{L_0^\lambda} z^{\alpha_0}) \tag{B.12}$$

of $M$ can be computed as

$$\chi_M = \prod_{j=1}^{\infty}(1 + zq^{\lambda+j-1})(1 + z^{-1}q^{-\lambda+j})\ , \tag{B.13}$$

where the coefficient of $z^k q^n$ is the number of different states in $M$ with charge $k$ and energy $n$.

On the other hand, one might as well choose a 'bosonic' basis of $M$: If $M$ is decomposed into a direct sum of eigenspaces of the charge operator $\alpha_0$ as $M = \bigoplus_{m\in\mathbb{Z}} M^m$ with the *m-th charged vacua* introduced as

$$|m\rangle = \psi^+_{-m}\cdots\psi^+_{-2}\psi^+_{-1}|0\rangle \quad \text{if } m \geq 0 \quad \text{and} \tag{B.14}$$

$$|m\rangle = \psi^-_{m}\cdots\psi^-_{-2}\psi^-_{-1}|0\rangle \quad \text{if } m \leq 0\ , \tag{B.15}$$

then

$$\{\alpha_{j_s}\cdots\alpha_{j_1}|m\rangle \mid 0 > j_1 \geq j_2 \geq \ldots\} \tag{B.16}$$

forms a 'bosonic' basis of $M^m$ with energy eigenvalues $\frac{1}{2}m^2 + (\lambda - \frac{1}{2})m + j_1 + \ldots + j_s$. Since the representation of the oscillator algebra consisting of the modes of the bosonic field is irreducible in each $M^m$, the bosonic character can be computed as

$$\chi_M = \sum_{m\in\mathbb{Z}} \frac{z^m q^{\frac{1}{2}m^2+(\lambda-\frac{1}{2})m}}{(q)_\infty} = q^{-\frac{(\lambda-\frac{1}{2})^2}{2}+\frac{1}{24}}\,\frac{\Theta_{\lambda-\frac{1}{2},\frac{1}{2}}}{\eta(q)}\ . \tag{B.17}$$

Equality of (B.13) and (B.17) results in

$$\prod_{j=1}^{\infty}(1-q^j)(1 + zq^{\lambda+j-1})(1 + z^{-1}q^{-\lambda+j}) = \sum_{m\in\mathbb{Z}} z^m q^{\frac{1}{2}m(m-1)+m\lambda}\ , \tag{B.18}$$

which reduces to just another form of the Jacobi triple product identity (3.2.4) [Jac29] (see also [And84, GR90]) for $\lambda = 0$. The Jacobi triple product identity is also a special case of the Cauchy identity or of another identity by Euler (see e.g. [Kir95, p. 36] or [And84, p. 19], respectively). As an aside, note that setting in (B.18) $\lambda = \frac{1}{3}$, $z = -1$ and replacing $q$ by $q^3$ results in the Euler identity

$$(q)_\infty = \prod_{j=1}^{\infty}(1-q^j) = \sum_{m\in\mathbb{Z}}(-1)^m q^{\frac{3}{2}m^2+\frac{1}{2}m}\ , \tag{B.19}$$

whereas setting $\lambda = \frac{1}{2}$, $z = -1$ and replacing $q$ by $q^2$ leads to the Gauss identity

$$\prod_{j=1}^{\infty}\frac{1+q^j}{1-q^j} = \sum_{m\in\mathbb{Z}} q^{m^2}\ . \tag{B.20}$$

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**Selbständigkeitserklärung**

Hiermit versichere ich, die vorliegende Diplomarbeit selbständig und unter ausschließlicher Verwendung der angegebenen Hilfsmittel angefertigt zu haben.

Hannover, den 16. Februar 2007

2. Juli 2007