ON SOLUTIONS OF NONCOMMUTATIVE YANG-MILLS THEORY IN FOUR DIMENSIONS

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Diploma Thesis at the
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June 26, 2002

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Chapter 1

Introduction

It has been suggested that at very small length scales position "measurements" might fail to commute. This fact can be described using a commutator relation between the coordinates, for instance:

$$[x^i, x^j] = \mathrm{i}\theta^{ij}, \tag{1.1}$$

where θ^{ij} are components of a constant tensor. Starting with a conventional field theory Lagrangian and interpreting the fields as being dependent on coordinates that satisfy the commutation relation (1.1), one can follow the usual development of perturbative quantum field theory with a few changes to define a large class of "noncommutative field theories" [1].

Noncommutative field theories have some problems. A consequence of postulating an uncertainty relation between position measurements is the non-locality of the theory. Noncommutativity of the coordinates also conflicts with Lorentz invariance. Despite these problems, the study of noncommutative field theories has some advantages. Namely, noncommutative theories might improve the renormalizability properties of a theory, although these theories could also have even a worse short distance behavior than conventional theories. It is believed that in quantum theories, including gravity, space-time could change its nature at distances comparable to the Planck scale. There are reasons to believe that any theory of quantum gravity will not be local in the conventional sense.

Another motivation for the study of noncommutative field theory is the discovery that a certain corner of string moduli space is described by noncommutative gauge theory [2]. String dynamics is modified nontrivially by a constant NS-NS two-form background in the presence of D-branes [3]. In this case, open strings ending on n coincident D-branes see a deformed space with metric $G_{\mu\nu}$ and acquire a noncommutativity parameter $\theta^{\mu\nu}$. This implies that the D-brane word volume carrying the U(n) Yang-Mills field becomes noncommutative [2, 3]. In the limit $\alpha' \to 0$ the string reduces to noncommutative U(n) gauge theory on the brane. The idea that for U(n) groups the open $N = 2$ string in 2+2 dimensions with a nonzero B-field coincides on tree level with noncommutative self-dual Yang-Mills is discussed in [4].

The dynamics of nonabelian gauge fields involves field configurations not accessible by perturbation theory of which instantons are the most prominent in the case of Euclidean space time. For the description of the nonperturbative structure of noncommutative gauge theory it is important to construct the noncommutative deformation of instanton configurations. The first examples of noncommutative instantons were constructed in [5]. It was shown that on noncommutative $\mathbb{R}^4$ nonsingular instantons exist even for the U(1) gauge group, i.e. that noncommutativity of the coordinates eliminates singular behavior of field configurations. These results initiated a lot of activity in this subject (see, e.g. [6]-[17]).

In this work we attempt to find some solutions to the noncommutative self-dual Yang-Mills equations. This thesis is structured as follows: In the second chapter the essential ideas regarding the Yang-Mills theory are introduced. The self-dual Yang-Mills equations, the Nahm equations and the concept of instanton are also presented here. In the third chapter the formalism of noncommutative field theory, and more specifically noncommutative gauge theory, are exposed. The star product formulation of noncommutative field theories, the operator formulation, and their equivalence through the Weyl transform is introduced. The fourth chapter is devoted to the derivation of two particular solutions of the self-dual Yang-Mills equations in noncommutative $\mathbb{R}^4_\theta$. In the last chapter a solution for the sigma model equation in noncommutative $\mathbb{R}^{2,0}_\theta$ is presented.

Chapter 2

Instantons

The ideas expressed in this chapter are based mainly on [18, 19], as well as on [20].

2.1 Yang-Mills fields

Consider a Lagrangian of the form

$$\mathcal{L}_o = \frac{1}{2}(\partial_\mu \phi) \cdot (\partial^\mu \phi) - V(\phi(x)). \tag{2.1}$$

Here $\phi(x) = [\phi_i(x), i = 1, ..., N]$ is a set of N-coupled scalar fields, x is a point in the Minkowskian space $\mathbb{R}^{1,3}$, the metric is $\mathrm{g}^{\mu\nu}$,

$$(\mathrm{g}^{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

with $\mu, \nu = 0, 1, 2, 3$, and $V(\phi(x))$ is the potential function. In the above expression for the Lagrangian $\mathcal{L}_o$ we have

$$(\partial_\mu \phi) \cdot (\partial^\mu \phi) = \sum_{\mu=0}^{3} \sum_{i=1}^{N} (\partial_\mu \phi_i)(\partial^\mu \phi_i).$$

A gauge potential $A_\mu(x)$ arises when trying to make the Lagrangian $\mathcal{L}_o$ invariant under the local transformation

$$\phi_i(x) \longrightarrow \phi_i(x)' = U_{ij}(x)\phi_j(x),$$

or suppressing the indices i and j

$$\phi(x) \longrightarrow \phi(x)' = U(x)\phi(x),$$

with $U(x) \in \mathrm{G}$ a Lie group. The matrix $U(x)$ can be written as

$$U(\theta) = \exp\left(J_a \varepsilon^a(x)\right).$$

Here the $\varepsilon^a(x)$ are arbitrary real functions of x and the J_a are the generators of G satisfying

$$[J_a, J_b] = f^c_{ab} J_c,$$

where f^c_{ab} are the totally antisymmetric real structure constants of the Lie group G, and $a, b, c, ... = 1, ..., \dim \mathrm{G}$. In order to construct a Lagrangian $\mathcal{L}$ which is invariant under the local transformation generated by U, the gauge potentials $A^a_\mu(x)$ (one for each generator of G) are introduced to form the gauge-covariant derivative

$$D_\mu = \left(\partial_\mu + g J_a A^a_\mu\right),$$

which replaces ∂_μ in equation (2.1). Here, g is the coupling constant. The Lagrangian $\mathcal{L}$ will remain invariant if $D_\mu\phi$ transforms as

$$D_\mu\phi \longrightarrow (D_\mu\phi)' = U D_\mu \phi.$$

This holds provided A^a_μ transforms in the following way

$$J_a A'^a_\mu = U(J_a A^a_\mu)U^{-1} - \frac{1}{g}(\partial_\mu U)U^{-1}. \tag{2.2}$$

If, for instance, we specify the group G to be U(2), then the structure constants f^c_{ab} become the completely antisymmetric tensor ϵ^c_{ab}, and we have the generators of U(2): $J_a = \sigma_a/2\mathrm{i}$, where σ_a are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.3}$$

The Pauli matrices satisfy the equation $\sigma_a\sigma_b = \delta_{ab} + \mathrm{i}\epsilon^c_{ab}\sigma_c$, and also have the property $\mathrm{Tr}[\sigma_a\sigma_b] = 2\delta_{ab}$, $a, b, c = 1, 2, 3$.

The gauge potential A_μ is defined as $A_\mu \equiv g A^a_\mu J_a$; it takes values in the Lie algebra of the group G. In this notation (2.2) can be written as $A'_\mu = U A_\mu U^{-1} - (\partial_\mu U) U^{-1}$. The Yang-Mills (YM) field strength tensor $F_{\mu\nu}$ is constructed using the matrix valued gauge potential A_μ as

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu],$$

or in component form

$$F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc} A^b_\mu A^c_\nu.$$

The relation between $F_{\mu\nu}$ and $F^a_{\mu\nu}$ is: $F_{\mu\nu} = g J_a F^a_{\mu\nu}$. From the fact that $F_{\mu\nu}$ can be written as

$$F_{\mu\nu} \equiv [D_\mu, D_\nu], \tag{2.4}$$

and from the transformation law $(D_\mu\phi)' = U D_\mu \phi$, one can see that the Yang-Mills field $F_{\mu\nu}$ transforms in the following way: $F'_{\mu\nu} = U F_{\mu\nu} U^{-1}$. This implies that the product $\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}] = \frac{1}{2} F^a_{\mu\nu} F^{a\mu\nu}$ is gauge invariant. It contains derivatives of A_μ, and therefore it becomes the Lagrangian $\mathcal{L}_{gauge} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$ for the potentials A_μ. When it is added to the Lagrangian $\mathcal{L}$, we obtain the complete gauge invariant Lagrangian $\mathcal{L}_{YM}$:

$$\mathcal{L}_{YM} = \mathcal{L} + \mathcal{L}_{gauge} = \frac{1}{2}(D_\mu\phi)\cdot(D^\mu\phi) - V(\phi(\mathbf{x})) - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}.$$

2.2 Euclidean Yang-Mills equations

To obtain the Euclidean version of a theory, the Euclidean metric $\delta^{\mu\nu}$ replaces the Minkowskian metric $\mathrm{g}^{\mu\nu}$. Also the space-time coordinate vector $(x^\mu)_{Min}$ $(\mu = 0, 1, 2, 3)$ is replaced by $(x^\mu)_{Euc}$ $(\mu = 1, 2, 3, 4)$. The Euclidean theory is left invariant under O(4) transformations rather than Lorentz transformations. Now the quantity

$$(x^\mu x^\mu)_{Euc} = \sum_{\mu=1}^{4} (x^\mu)^2_{Euc},$$

is left invariant, and any other Lorentz tensor is replaced by an O(4) tensor.

In the absence of matter fields $(\phi \to 0)$, the action for the Euclidean YM theory becomes

$$S = \int d^4x \left(-\frac{1}{2}\mathrm{Tr}[F_{\mu\nu}F_{\mu\nu}] \right) = \int d^4x \left(-\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} \right).$$

The extrema of the action S results in the Yang-Mills equations of motion:

$$\partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] \equiv [D_\mu, F_{\mu\nu}] = 0, \tag{2.5}$$

or in component form:

$$\partial_\mu F^a_{\mu\nu} + g\epsilon^{abc}A^b_\mu F^c_{\mu\nu} = 0.$$

Now, consider the Jacobi identity for D_μ:

$$[D_\lambda, [D_\mu, D_\nu]] + [D_\mu, [D_\nu, D_\lambda]] + [D_\nu, [D_\lambda, D_\mu]] \equiv 0. \tag{2.6}$$

Using the definition (2.4) and multiplying the equation (2.6) by $\epsilon_{\mu\nu\rho\sigma}$, we obtain

$$[D_\mu, \tilde{F}_{\mu\nu}] \equiv 0, \tag{2.7}$$

where

$$\tilde{F}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}F_{\lambda\rho}, \tag{2.8}$$

is the dual of the gauge field strength tensor $F_{\mu\nu}$. From equation (2.7) it follows that any gauge field which is self-dual, $F_{\mu\nu} = \tilde{F}_{\mu\nu}$, automatically satisfies the equations of motion (2.5). The (anti-)self-dual Yang-Mills equations are defined as:

$$\pm F_{\mu\nu} = \tilde{F}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}F_{\lambda\rho}. \tag{2.9}$$

The equations (2.9) are first order non-linear partial differential equations for the gauge potential A_μ, and therefore simpler than equations (2.5), which are second order partial differential equations.

2.3 Nahm's equations

Nahm's equations had appeared in the construction of monopole solutions to the Bogomolny equations [21]. Nahm's equations are also obtained from the self-dual Yang-Mills (SDYM) equations in $\mathbb{R}^4$ when the gauge potentials A_μ are considered as independent of the coordinates x_2, x_3 and x_4. With these restrictions, the SDYM equations in $\mathbb{R}^{4,0}$ can be reduced to the following equations:

$$\begin{aligned}
\dot{T}_1 &= [T_2, T_3] + [T_4, T_1], \\
\dot{T}_2 &= [T_3, T_1] + [T_4, T_2], \\
\dot{T}_3 &= [T_1, T_2] + [T_4, T_3],
\end{aligned} \tag{2.10}$$

which are called Nahm's equations. Here

$$\begin{aligned}
&T_1 \equiv -A_2, \quad T_2 \equiv -A_3, \quad T_3 \equiv -A_4, \quad T_4 \equiv -A_1, \\
&\dot{T}_a \equiv dT_a/d\tau, \quad \tau \equiv x_1.
\end{aligned}$$

Consider the following ansatz for the fields A_μ:

$$A_\mu = 2\eta^a_{\mu\nu} x_\nu T_a(\tau), \tag{2.11}$$

where the Lie algebra valued functions T_a depend only on $\tau = x_\mu x^\mu$, and $\eta^a_{\mu\nu}$ are the components of the self-dual t'Hooft tensor defined as:

$$\eta^a_{bc} = \epsilon^a_{bc}, \quad \eta^a_{\mu 4} = \delta^a_\mu, \quad \eta^a_{\mu\nu} = -\eta^a_{\nu\mu},$$

with $\mu, \nu = 1, ..., 4;\ a, b = 1, 2, 3$. If the fields A_μ are replaced with the ansatz (2.11) then the Euclidean SDYM equations:

$$F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho} F_{\lambda\rho}, \tag{2.12}$$

are reduced to Nahm's equations (2.10) with $\tau = x_\mu x^\mu$ [22]. To each solution of the equations (2.10) obtained from the substitution of the ansatz (2.11) in (2.12) it corresponds a solution of the self-duality equation (2.12) of the form (2.11).

2.4 Instantons

Instantons are finite-action localized solutions to the Euclidean field equations of a theory. The Yang-Mills instantons are finite-action solutions to the equations (2.5). These finite-action configurations $A_\mu(x)$ can be classified into a discrete infinity of homotopy classes. This homotopy classification is based on the boundary conditions which $A_\mu(x)$ must fulfill in order to have a finite action. Each class can be characterized by an integer Q, called the Pontryagin index.

A field $A_\mu(x)$ belonging to a given sector Q cannot be continuously deformed to another field $A'_\mu(x)$ with an index $Q' \neq Q$ without violating the finiteness of the action. For a given configuration $A_\mu(x)$, this index Q can be found as

$$Q = -\frac{1}{16\pi^2}\int d^4x \,\mathrm{Tr}[\tilde{F}_{\mu\nu}F_{\mu\nu}]. \tag{2.13}$$

The index Q is also called topological charge. In the corresponding quantum field theory the topological charge plays the role of a quantum number.

A particular case of finite-action configuration is $S = 0$. The action S vanishes if and only if $F_{\mu\nu} = 0$. This condition is satisfied by $A_\mu = 0$, but due to the fact $F_{\mu\nu} = 0$ is a gauge invariant condition, also "pure gauge" fields:

$$A_\mu(x) = U(x)\partial_\mu U^{-1}(x) \tag{2.14}$$

will lead to $S = 0$.

Finite action-configurations with $S \neq 0$ require $F_{\mu\nu}$ to vanish in the three-dimensional spherical surface at $r = \infty$, denoted as S_3^{phy}, where $r \equiv |x| = (x_1^2 + x_2^2 + x_3^2 + x_4^2)^{1/2}$. Also $F_{\mu\nu}$ must fall to zero faster than $1/r^2$ as $r \to \infty$. If $F_{\mu\nu}$ will vanishes when $r \to \infty$ then A_μ has to take the form (2.14); in this case the following condition must apply:

$$\lim_{r\to\infty} A_\mu(x) = \lim_{r\to\infty} U(x)\partial_\mu U^{-1}(x). \tag{2.15}$$

Now, in the limit $r \to \infty$, the function U will depend on 3 parameters which can be for example the polar angles α_1, α_2 and α_3 in four dimensions. The condition (2.15) can be written as

$$A_\mu(x)_{S_3^{phy}} = U(\alpha_1, \alpha_2, \alpha_3)\partial_\mu U^{-1}(\alpha_1, \alpha_2, \alpha_3). \tag{2.16}$$

The function $U(\alpha_1, \alpha_2, \alpha_3)$ represents a mapping of S_3^{phy} into the group space of U(2). The matrices U can be expressed as:

$$U = \sum_{\mu=1}^{4} a_\mu s_\mu,$$

where $s_4 = \mathbf{1}_2$ the unit 2×2 matrix, $s_{1,2,3} = \mathrm{i}\sigma_{1,2,3}$ and a_μ are any real numbers which satisfy

$$\sum_{\mu} a_\mu a_\mu = 1. \tag{2.17}$$

That is, the group U(2) can be parameterized by the four variables a_μ which satisfy (2.17) and therefore the group space is the three-dimensional surface S_3^{int} of the unit sphere in four dimensions.

The mappings of $S_3^{phy} \to S_3^{int}$ can be divided into a discrete infinity of homotopy classes, characterized by the index Q defined in (2.13). One can think in Q as the "winding number" of S_3^{phy} into S_3^{int}, or in other words, the number of times the group space of U(2) is wrapped around S_3^{phy}. The finite configurations A_μ can be characterized by the same index Q as the corresponding gauge function U. Any given sector corresponds to a given homotopy class of the $U(\alpha_1, \alpha_2, \alpha_3)$'s occurring in the boundary condition (2.16) for A_μ.

The fact that the topological charge Q can be seen as the "winding number" of S_3^{phy} into S_3^{int} is illustrated now. The integral (2.13) can be written as a surface integral over S_3^{phy}, and then, the fields A_μ are expressed in the form (2.16), according to their asymptotic behavior. The resulting surface integral depends now only on the function U. Finally, this integral reduces to an integral over the group measure. A result from homotopy theory is that when we integrate over S_3^{phy} once, then the group space S_3^{int} may be spanned an integral number of times. In this way, Q will be proportional to this integer. A more detailed demonstration is given in [19].

The first step of the procedure described above, namely the transformation of the integral (2.13) into a surface integral is accomplished using Gauss' theorem. For this, the integrand has to be expressed as a divergence. Using the cyclic property of the trace and the identity:

$$\begin{aligned} D_\mu \tilde{F}_{\mu\nu} &= \partial_\mu \tilde{F}_{\mu\nu} + [A_\mu, \tilde{F}_{\mu\nu}] \\ &= \epsilon_{\mu\nu\alpha\beta}\{\partial_\mu(\partial_\alpha A_\beta + A_\alpha A_\beta) + [A_\mu, (\partial_\alpha A_\beta + A_\alpha A_\beta)]\} \\ &= 0, \end{aligned}$$

the integrand (2.13) can be written as:

$$\mathrm{Tr}[F_{\mu\nu}\tilde{F}_{\mu\nu}] = \mathrm{Tr}\{\epsilon_{\mu\nu\alpha\beta}[2\partial_\mu(A_\nu \partial_\alpha A_\beta + \frac{2}{3}A_\nu A_\alpha A_\beta)]\}. \tag{2.18}$$

Thus

$$\mathrm{Tr}[\tilde{F}_{\mu\nu}F_{\mu\nu}] = -\frac{1}{16\pi^2}\partial_\mu j_\mu,$$

where

$$j_\mu = -\frac{1}{8\pi^2}\epsilon_{\mu\nu\alpha\beta}\mathrm{Tr}[A_\nu(\partial_\alpha A_\beta + \frac{2}{3}A_\alpha A_\beta)].$$

Gauss' theorem can be used now:

$$-\frac{1}{16\pi^2}\int d^4x \mathrm{Tr}[F_{\mu\nu}\tilde{F}_{\mu\nu}] = \oint_{S_3^{phy}} d\sigma_\mu j_\mu.$$

On S_3^{phy} we have $F_{\alpha\beta}$=0, that means $\epsilon_{\mu\nu\alpha\beta}\partial_\alpha A_\beta = -\epsilon_{\mu\nu\alpha\beta}A_\alpha A_\beta$, and therefore:

$$Q = \frac{1}{24\pi^2}\oint_{S_3^{phy}} d\sigma_\mu\, \epsilon_{\mu\nu\alpha\beta}\, \mathrm{Tr}[A_\nu A_\alpha A_\beta].$$

At the end, using (2.16), we have an expression for Q in terms of U:

$$Q = -\frac{1}{24\pi^2}\oint_{S_3^{phy}} d\sigma_\mu\, \epsilon_{\mu\nu\alpha\beta}\, \mathrm{Tr}[(\partial_\nu U)U^{-1}(\partial_\alpha U)U^{-1}(\partial_\beta U)U^{-1}]. \tag{2.19}$$

The hyper-sphere S_3^{int} can be parameterized by three independent variables ξ_1, ξ_2 and ξ_3. A given group element U corresponds to some point (ξ_1, ξ_2, ξ_3). The group measure around the element U may be written as

$$d\mu(U) = \rho(\xi_1, \xi_2, \xi_3,)d\xi_1 d\xi_2 d\xi_3,$$

where

$$\rho(\xi_1, \xi_2, \xi_3) = \epsilon_{ijk}\mathrm{Tr}\left(U^{-1}\frac{\partial U}{\partial \xi_i}U^{-1}\frac{\partial U}{\partial \xi_j}U^{-1}\frac{\partial U}{\partial \xi_k}\right).$$

The group measure $d\mu(U)$ is left invariant by group translations

$$U \to U' = \tilde{U}U,$$

where $\tilde{U}$ is a fixed group element. That means

$$d\mu(U) = d\mu(U'),$$

or

$$\rho(\xi_1, \xi_2, \xi_3,)d\xi_1 d\xi_2 d\xi_3 = \rho(\xi_1', \xi_2', \xi_3',)d\xi_1' d\xi_2' d\xi_3'.$$

In (2.19) the matrices U may be considered as a function of ξ_i, and the ξ_i are functions of the spatial coordinates x_μ. Therefore (2.19) can be expressed as

$$-\frac{1}{24\pi^2}\oint_{S_3^{phy}} d\sigma_\mu \left[\epsilon_{\mu\nu\alpha\beta}\mathrm{Tr}\left(U^{-1}\frac{\partial U}{\partial \xi_i}U^{-1}\frac{\partial U}{\partial \xi_j}U^{-1}\frac{\partial U}{\partial \xi_k}\right)\frac{\partial \xi_i}{\partial x_\nu}\frac{\partial \xi_j}{\partial x_\alpha}\frac{\partial \xi_k}{\partial x_\beta}\right].$$

It is possible to distort the hyper-sphere S_3^{phy} into the surface of an infinite hypercube whose sides are the surfaces $x_\mu = \pm\infty$. By taking any surface on the hypercube, for instance $x_4 = -\infty$ and integrating over it one gets the contribution

$$-\frac{1}{24\pi^2}\int dx_1 dx_2 dx_3\, \epsilon_{mnl}\,\mathrm{Tr}\left(U^{-1}\frac{\partial U}{\partial \xi_i}U^{-1}\frac{\partial U}{\partial \xi_j}U^{-1}\frac{\partial U}{\partial \xi_k}\right)\frac{\partial \xi_i}{\partial x_m}\frac{\partial \xi_j}{\partial x_n}\frac{\partial \xi_k}{\partial x_l},$$

where l, m, n range from 1 to 3. But we have

$$\epsilon_{mnl}\frac{\partial \xi_i}{\partial x_m}\frac{\partial \xi_j}{\partial x_n}\frac{\partial \xi_k}{\partial x_l}dx_1 dx_2 dx_3 = \epsilon_{ijk} d\xi_1 d\xi_2 d\xi_3.$$

In this way the contribution from the surface $x_4 = -\infty$ to the integral (2.19) is

$$-\frac{1}{24\pi^2}\int d\xi_1 d\xi_2 d\xi_3\, \epsilon_{ijk}\,\mathrm{Tr}\left(U^{-1}\frac{\partial U}{\partial \xi_i}U^{-1}\frac{\partial U}{\partial \xi_j}U^{-1}\frac{\partial U}{\partial \xi_k}\right). \tag{2.20}$$

This integral is proportional to an integral over the group measure $d\mu(U)$. Each one of the surfaces of the hypercube contribute in a similar way, then

$$Q \propto \int d\xi_1 d\xi_2 d\xi_3\, \rho(\xi_1, \xi_2, \xi_3) \propto \int d_\mu(U).$$

The topological charge which originally was written as a function of $F_{\mu\nu}$ reduces to an integral over the group space of the group measure.

The Yang-Mills equations (2.5) are obtained by extremising the action S through the variational principle

$$\delta S[A_\mu] = 0.$$

Self-dual and anti-self-dual configurations extremise the action S and solve the Yang-Mills equations (2.5). To see this, one employs the identity

$$-\int d^4x\,\mathrm{Tr}[(F_{\mu\nu}+\tilde{F}_{\mu\nu})^2] \geq 0.$$

Using the property

$$\mathrm{Tr}[F_{\mu\nu}F_{\mu\nu}] = \mathrm{Tr}[\tilde{F}_{\mu\nu}\tilde{F}_{\mu\nu}],$$

the following inequality for the action is obtained:

$$-\int d^4x\,\mathrm{Tr}[F_{\mu\nu}F_{\mu\nu}] \geq \mp\int d^4x\,\mathrm{Tr}[\tilde{F}_{\mu\nu}F_{\mu\nu}],$$

or

$$S \geq (8\pi^2/g^2)|Q|.$$

Hence, the equality, and therefore the absolute minimum value of S in a given sector Q occurs when equations (2.9) take place.

2.5 't Hooft Instantons

A multi-instanton solution to the SDYM equations (2.12) can be obtained with the aid of the 't Hooft ansatz:

$$A_\mu(x) = \mathrm{i}\overline{\Sigma}_{\mu\nu}\partial_\nu(\ln\phi(x)). \tag{2.21}$$

The equation on the scalar function $\phi(x)$ has to be obtained after inserting the ansatz (2.21) in equations (2.12), and $\overline{\Sigma}_{\mu\nu}$ is defined as

$$\overline{\Sigma}_{\mu\nu} = \frac{1}{2}\begin{bmatrix} 0 & +\sigma_3 & -\sigma_2 & -\sigma_1 \\ -\sigma_3 & 0 & +\sigma_1 & -\sigma_2 \\ +\sigma_2 & -\sigma_1 & 0 & -\sigma_3 \\ +\sigma_1 & +\sigma_2 & +\sigma_3 & 0 \end{bmatrix}, \tag{2.22}$$

where the σ_i are the Pauli matrices (2.3). A compact form to write $\overline{\Sigma}_{\mu\nu}$ is

$$\overline{\Sigma}_{\mu\nu} = \bar{\eta}^a_{\mu\nu}\sigma_a/2; \qquad a = 1,2,3,$$

where the anti-self-dual 't Hooft tensor $\bar{\eta}^a_{\mu\nu}$ is defined as:

$$\bar{\eta}^a_{\mu\nu} = -\bar{\eta}^a_{\nu\mu} = \begin{cases} \epsilon^a_{\mu\nu} & \text{for } \mu,\nu = 1,2,3 \\ -\delta^a_\mu & \text{for } \nu = 4. \end{cases} \tag{2.23}$$

Apart from antisymmetry, the matrix $\overline{\Sigma}_{\mu\nu}$ has also the following properties:

$$[\overline{\Sigma}_{\mu\sigma}, \overline{\Sigma}_{\nu\rho}] = \mathrm{i}[\delta_{\mu\nu}\overline{\Sigma}_{\sigma\rho} + \delta_{\rho\sigma}\overline{\Sigma}_{\mu\nu} - \delta_{\mu\rho}\overline{\Sigma}_{\sigma\nu} - \delta_{\nu\sigma}\overline{\Sigma}_{\mu\rho}],$$

$$\epsilon_{\mu\nu\alpha\beta}\,\overline{\Sigma}_{\beta\sigma} = [\delta_{\mu\sigma}\overline{\Sigma}_{\nu\alpha} + \delta_{\nu\sigma}\overline{\Sigma}_{\alpha\mu} + \delta_{\alpha\sigma}\overline{\Sigma}_{\mu\nu}],$$

and anti-self-duality

$$\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\overline{\Sigma}_{\alpha\beta} = -\overline{\Sigma}_{\mu\nu}.$$

To get the form of the function $\phi(x)$, the ansatz (2.21) is inserted in the field tensor $F_{\mu\nu}$ to get

$$\begin{aligned} F_{\mu\nu} = \;& \mathrm{i}\overline{\Sigma}_{\nu\sigma}(\partial_\mu\partial_\sigma(\ln\phi) - \partial_\mu(\ln\phi)\partial_\sigma(\ln\phi)) \\ & -\mathrm{i}\overline{\Sigma}_{\mu\sigma}(\partial_\nu\partial_\sigma(\ln\phi) - \partial_\nu(\ln\phi)\partial_\sigma(\ln\phi)) \\ & -\mathrm{i}\overline{\Sigma}_{\mu\nu}(\partial_\sigma(\ln\phi))^2. \end{aligned}$$

In a similar way $\tilde{F}_{\mu\nu}$ takes the form

$$\begin{aligned} \tilde{F}_{\mu\nu} &= \mathrm{i}\epsilon_{\mu\nu\alpha\beta}\,[\overline{\Sigma}_{\beta\sigma}(\partial_\alpha\partial_\sigma(\ln\phi) - \partial_\alpha(\ln\phi)\partial_\sigma(\ln\phi)) \\ &\quad -\tfrac{1}{2}\overline{\Sigma}_{\alpha\beta}(\partial_\sigma(\ln\phi))^2], \\ &= -\mathrm{i}\overline{\Sigma}_{\nu\alpha}(\partial_\alpha\partial_\mu(\ln\phi) - \partial_\alpha(\ln\phi)\partial_\mu(\ln\phi)) - (\mu\leftrightarrow\nu) \\ &\quad +\mathrm{i}\overline{\Sigma}_{\mu\nu}\partial_\sigma\partial_\sigma(\ln\phi). \end{aligned}$$

The self-duality condition gives as a result

$$\partial_\mu\partial_\sigma(\ln\phi) - \partial_\mu(\ln\phi)\partial_\sigma(\ln\phi) = \partial_\sigma\partial_\mu(\ln\phi) - \partial_\sigma(\ln\phi)\partial_\mu(\ln\phi) \tag{2.24}$$

and

$$\partial_\sigma\partial_\sigma(\ln\phi) - (\partial_\sigma(\ln\phi))^2 = 0. \tag{2.25}$$

The equation (2.25) can be written as

$$\Box\phi/\phi = 0 \tag{2.26}$$

where $\Box\phi$ in Euclidean metric is $\partial_\sigma\partial_\sigma\phi$. Singular non-trivial solutions to the equation (2.26) lead to interesting and non-singular solutions for the gauge field. Consider the function

$$\phi(x) = 1/|x|^2.$$

This function solves (2.26). It is possible to generalize this solution to

$$\phi(x) = 1 + \sum_{i=1}^{N} \frac{\lambda_i^2}{|x_\mu - a_{i\mu}|^2}. \tag{2.27}$$

For $N = 1$, using $y_\mu \equiv (x_\mu - a_{1\mu})$ and $y^2 \equiv y_\mu y_\mu$ we get

$$\phi(x) = 1 + \lambda_1^2/y^2,$$

which results in

$$A_\mu(x) = -2\mathrm{i}\lambda_1^2\,\overline{\Sigma}_{\mu\nu}\frac{y_\nu}{y^2(y^2+\lambda_1^2)}.$$

This solution is singular at $y = 0$. But transforming $A_\mu(x)$ with the matrix:

$$U_1(x) = (x_4 + \mathrm{i}x_j\sigma_j)/|x| = \sum_{\mu=1}^{4} \hat{x}_\mu s_\mu,$$

this singularity can be removed. Using the relation

$$[U_1(y)]^{-1}\frac{\partial}{\partial y_\mu}[U_1(y)] = -2\mathrm{i}\overline{\Sigma}_{\mu\nu}(y_\nu/y^2),$$

one can transform $A_\mu(x)$ with $U_1(y)$ to get

$$A_\mu(x) \to A'_\mu(x) = U_1(\partial_\mu U_1^{-1})\frac{y^2}{y^2+\lambda_1^2}.$$

Now, the tensor $\Sigma_{\mu\nu}$ is defined by

$$\Sigma_{\mu\nu} = \eta^a_{\mu\nu}\sigma_a/2,$$

with

$$\eta^a_{\mu\nu} = -\eta^a_{\nu\mu} = \begin{cases} \epsilon^a_{\mu\nu} & \text{for } \mu,\nu = 1,2,3 \\ -\delta^a_\mu & \text{for } \nu = 4. \end{cases}$$

It satisfies the relation

$$[U_1(y)]\frac{\partial}{\partial y_\mu}[U_1(y)]^{-1} = -2\mathrm{i}\Sigma_{\mu\nu}(y_\nu/y^2),$$

Then, the gauge transformed solution becomes

$$A'_\mu(x) = -2\mathrm{i}\Sigma_{\mu\nu}\frac{y_\nu}{y^2+\lambda_1^2} = -2\mathrm{i}\Sigma_{\mu\nu}\frac{(x-a_1)_\nu}{|(x-a_1)|^2+\lambda_1^2}. \tag{2.28}$$

This 1-instanton solution has no singularities provided $\lambda_1 \neq 0$. The corresponding field tensor is

$$F'_{\mu\nu}(x) = 4\mathrm{i}\Sigma_{\mu\nu}\frac{\lambda_1^2}{[|x-a_1|^2+\lambda_1^2]^2}.$$

The gauge potential (2.28) is a solution of the Euclidean Yang-Mills equation (2.5) because $F'_{\mu\nu}$ is self-dual, that means, $F'_{\mu\nu}$ minimize the action. If we calculate the Pontryagin index Q for this solution, then the value $Q = +1$ is obtained.

As in the case of the 1-instanton solution, it is possible to obtain an N-instanton solution by inserting the function (2.27) into (2.21). Then one gets

$$A_\mu(x) = \mathrm{i}\overline{\Sigma}_{\mu\nu}\partial_\nu\left[\ln\left(1+\sum_{i=1}^{N}\frac{\lambda_i^2}{|\lambda_i|^2}\right)\right] = -2\mathrm{i}\overline{\Sigma}_{\mu\nu}\frac{\displaystyle\sum_{i=1}^{N}\frac{\lambda_i^2 y_{i\nu}}{|y_i|^4}}{1+\displaystyle\sum_{j=1}^{N}\frac{\lambda_j^2}{|\lambda_j|^2}} \tag{2.29}$$

where $(y_i)_\mu \equiv (x-a_i)_\mu$; $i,j = 1,2,...,N$. Now, the field (2.29) has N singularities at $x = a_i$; $i = 1,...,N$. These singularities can be removed in a similar way as it was done in the 1-instanton case by transforming $A_\mu(x)$ with the gauge function

$$U_N(x) = (Z_4 + \mathrm{i}\mathbf{Z}\cdot\sigma)/\sqrt{Z^2},$$

with

$$Z_\mu(x) = \sum_{i=1}^{N}\beta_i\left(\frac{y_{i\mu}}{y_i^2}\right); \quad y_i = x - a_i,$$

where β_i are real constants. Then we have

$$A'_\mu(x) = U_N(x)(A_\mu + \partial_\mu)U_N^{-1}(x).$$

When x approaches a_i we obtain

$$
\begin{aligned}
A'_\mu(x) \quad \underset{x \to a_i}{\longrightarrow} \quad & U_1(y_i)[U_1^{-1}(y_i)\partial_\mu U_1(y_i)]U_1^{-1}(y_i) \\
& + U_1^{-1}(y_i)\partial_\mu U_1(y_i) + \text{possible finite terms} \\
= \quad & \partial_\mu[U_1(y_i)U_1^{-1}(y_i)] + \text{finite terms.}
\end{aligned}
$$

After some calculations, it can be seen that the first term vanishes and the singularity at $x = a_i$ is removed. However $U_N(x)$ has still singularities at $y_{i\mu} = 0$ and at $Z_\mu = 0$, but it is possible to choose the constants β_i in such a way that Z_μ never vanishes, and therefore the gauge transformed field $A'_\mu(x)$ becomes non-singular (see [19] p.107).

Finally, this N-instanton solution $A'_\mu(x)$ does correspond to a Pontryagin index $Q = N$. Intuitively, one can think in $A'_\mu(x)$ as a collection of N single instantons.

Chapter 3

Noncommutative gauge field theory

3.1 Gauge transformation

A discussion of noncommutative field theories can be found in [23], [24], [3] and [1]. In noncommutative geometry the coordinates $\hat{x}^i$ do not commute with the gauge transformations. In [25] a special formulation of noncommutative gauge theory is exposed. The starting point is the definition of a gauge potential and its corresponding transformation law, such that adding this potential to the coordinates $\hat{x}^i$ make them covariant.

Consider the associative algebraic structure $\mathcal{A}_x$ defined in terms of a set of generators $\hat{x}^i$ (coordinates) and relations:

$$[\hat{x}^i, \hat{x}^j] = \mathrm{i}\theta^{ij}, \qquad \theta^{ij} \in \mathbb{C}. \tag{3.1}$$

Also consider fields ψ as elements of the algebra $\mathcal{A}_x$:

$$\psi(\hat{x}) = \psi(\hat{x}^1, ..., \hat{x}^N) \in \mathcal{A}_x. \tag{3.2}$$

An infinitesimal gauge transformation $\delta\psi$ of the field ψ can be written in the form

$$\delta\psi(\hat{x}) = \mathrm{i}\alpha(\hat{x})\psi(\hat{x}), \qquad \alpha(\hat{x}), \psi(\hat{x}) \in \mathcal{A}_x. \tag{3.3}$$

This is called a covariant transformation law of a field. Since $\alpha(\hat{x}) \in \mathcal{A}_x$ then (3.3) is the equivalent of an abelian gauge transformation. In the case $\alpha(\hat{x}) \in M_n(\mathcal{A}_x)$, the matrices with elements in $\mathcal{A}_x$, eq. (3.3) is the equivalent of a nonabelian gauge transformation.

If a field ψ is multiplied on the left by a coordinate $\hat{x}^i$ and then transformed, we get:

$$\delta(\hat{x}^i\psi) = \mathrm{i}\hat{x}^i\alpha(\hat{x})\psi \neq \mathrm{i}\alpha(\hat{x})\hat{x}^i\psi. \tag{3.4}$$

Here the coordinates are considered invariant under the action of a gauge transformation:

$$\delta\hat{x}^i = 0.$$

From (3.4) one can notice that multiplication on the left by $\hat{x}^i$ is not a covariant operation, but it is possible to introduce a covariant coordinate $\hat{X}^i$ in such a way that

$$\delta(\hat{X}^i\psi) = \mathrm{i}\alpha\hat{X}^i\psi, \tag{3.5}$$

and this implies:

$$\delta(\hat{X}^i) = \mathrm{i}[\alpha, \hat{X}^i].$$

An ansatz for $\hat{X}^i$ of the form:

$$\hat{X}^i = \hat{x}^i + A^i(\hat{x}), \quad A^i(\hat{x}) \in \mathcal{A}_x. \tag{3.6}$$

will satisfy equation (3.5) provided $A^i(\hat{x})$ transforms as:

$$\delta A^i = \mathrm{i}[\alpha, A^i] - \mathrm{i}[\hat{x}^i, \alpha]. \tag{3.7}$$

The covariant tensor T^{ij} is defined in the following way:

$$T^{ij} = [\hat{X}^i, \hat{X}^j] - \mathrm{i}\theta^{ij}. \tag{3.8}$$

Using equation (3.6) T^{ij} can be written as

$$T^{ij} = [A^i, \hat{x}^j] + [\hat{x}^i, A^j] + [A^i, A^j]. \tag{3.9}$$

From the relation:

$$\delta T^{ij} = [\delta A^i, \hat{x}^j] + [\hat{x}^i, \delta A^j] + [\delta A^i, A^j] + [A^i, \delta A^j], \tag{3.10}$$

and the Jacobi identity, the transformation law for T^{ij} is obtained:

$$\delta T^{ij} = \mathrm{i}[\alpha, T^{ij}]. \tag{3.11}$$

That means, T^{ij} is a covariant tensor.

3.2 The Weyl transform

It is possible to associate the elements of $\mathcal{A}_x$ with functions f of classical variables using the prescription known as the Weyl transform:

$$W(f) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int d^n k \, e^{ik_j \hat{x}^j} \tilde{f}(k), \tag{3.12}$$

where $\tilde{f}(k)$ is the Fourier transform of f:

$$\tilde{f}(k) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int d^n x \, e^{-ik_j x^j} f(x). \tag{3.13}$$

The $\hat{x}^i$ are elements of $\mathcal{A}_x$ and the x^i are the associated classical variables. Now, the Moyal-Weyl product, or star product defined as:

$$f \star g(x) = \exp\left(\frac{\mathrm{i}}{2} \frac{\partial}{\partial x^i} \theta^{ij} \frac{\partial}{\partial y^j}\right) f(x) g(y) \bigg|_{y \to x}, \tag{3.14}$$

has the following property:

$$W(f)W(g) = W(f \star g). \tag{3.15}$$

Namely, the Weyl transform takes operator multiplication into the star product of functions of classical variables. In particular the commutator (3.1) is obtained from the ordinary commutator $[x^i, x^j]$ by replacing the ordinary commutative product with the star product:

$$\begin{aligned} &\textit{commutative}: && x^i x^j - x^j x^i = 0, \\ &\textit{noncommutative}: && x^i \star x^j - x^j \star x^i = \mathrm{i}\theta^{ij}, \end{aligned} \tag{3.16}$$

and then applying the Weyl transform. Here we used the fact that under the Weyl transform:

$$\hat{x}^i \to x^i. \tag{3.17}$$

If a field is represented by a classical function $\psi(x)$, then the gauge transformation (3.3) represented by the classical function $\alpha(x)$ takes the form

$$\delta_\alpha \psi(x) = \mathrm{i}\alpha(x) \star \psi(x), \tag{3.18}$$

and the transformation law of $A^i(x)$ (which represents the element $A^i \in \mathcal{A}_x$) is

$$\delta A^i = \mathrm{i}[\alpha, A^i]_\star - \mathrm{i}[\hat{x}^i, \alpha]_\star, \tag{3.19}$$

where $[a, b]_\star$ means that in the commutator the ordinary product is replaced by the $\star$ product, just as in (3.16). In a similar way the tensor T^{ij} transforms as:

$$\delta T^{ij} = \mathrm{i}[\alpha, T^{ij}]_\star. \tag{3.20}$$

3.3 Noncommutative gauge theory

The derivative operator which acts on elements of $\mathcal{A}_x$ takes the form of a commutator, specifically:

$$[\hat{x}^i, f(\hat{x})] = \mathrm{i}\theta^{ij}\partial_j f(\hat{x}). \tag{3.21}$$

This can be seen as follows:

$$\begin{aligned}
[\hat{x}^i, f] &= \hat{x}^i f - f\hat{x}^i, \\
&= W(x^i)W(f) - W(f)W(x^i), \\
&= W(x^i \star f - f \star x^i), \\
&= W(\mathrm{i}\theta^{ij}\partial_j f), && (3.22) \\
&= \mathrm{i}\theta^{ij}\partial_j f(\hat{x}). && (3.23)
\end{aligned}$$

Here we used the fact that: $x^i \star f - f \star x^i = \theta^{ij}\partial_j f$, which follows from (3.14). In (3.22), $\partial_j f$ is the ordinary derivative of the classical function f, while in (3.23) $\partial_j f(\hat{x})$ is a derivative which acts on elements $f(\hat{x}) \in \mathcal{A}_x$.

The derivative (3.21) has the property of linearity: $\partial(\mathrm{a}f + \mathrm{b}f) = \mathrm{a}\partial f + \mathrm{b}\partial g$, satisfies Leibnitz's rule: $\partial(fg) = \partial(f)g + f\partial(g)$, and also: $\partial_j \hat{x}^i = \delta^i_j$.

Considering equation (3.21), we can write the transformation law (3.7) of a gauge potential A^i as:

$$\delta A^i = \theta^{ij}\partial_j a + \mathrm{i}[\alpha, A^i]. \tag{3.24}$$

Expressing the gauge potential $\hat{A}$ of noncommutative Yang-Mills theory as:

$$A^i \equiv \theta^{ij} \hat{A}_j, \tag{3.25}$$

we obtain the transformation law for the gauge field $\hat{A}$:

$$\delta \hat{A}_j = \partial_j \alpha + \mathrm{i}[\alpha, \hat{A}_j]. \tag{3.26}$$

In a similar way, the field strength $\hat{F}$ written in terms of T^{ij} is:

$$T^{ij} = \mathrm{i}\theta^{ik}\theta^{jl}\hat{F}_{kl}, \tag{3.27}$$

and from this it follows that the field strength $\hat{F}$ has the form:

$$\hat{F}_{kl} = \partial_k \hat{A}_l - \partial_l \hat{A}_k - \mathrm{i}[\hat{A}^k, \hat{A}^l]. \tag{3.28}$$

The field strength $\hat{F}$ has the following transformation law:

$$\delta \hat{F}_{kl} = \mathrm{i}[\alpha, \hat{F}_{kl}]. \tag{3.29}$$

The relation of the formulas exposed in this section with the noncommutative abelian Yang-Mills theory can be appreciated better if the elements of $\mathcal{A}_x$ are represented by functions of the classical variables x^i, and the star product is used:

$$\begin{aligned}
\delta \hat{A}_j &= \partial_j \alpha + \mathrm{i}\alpha \star \hat{A}_j - \mathrm{i}\hat{A}_j \star \alpha, \\
\hat{F}_{kl} &= \partial_k \hat{A}_l - \partial_l \hat{A}_k - \mathrm{i}\hat{A}^k \star \hat{A}^l + \mathrm{i}\hat{A}^l \star \hat{A}^k, \\
\delta \hat{F}_{kl} &= \mathrm{i}\alpha \star \hat{F}_{kl} - \mathrm{i}\hat{F}_{kl} \star \alpha.
\end{aligned}$$

From this one can see that if the noncommutative star product is replaced by the ordinary commutative product ($\theta^{ij} \to 0$), then we obtain the ordinary commutative abelian Yang-Mills theory.

3.4 Gauge groups in noncommutative gauge theory

In the case of nonabelian gauge theories, the parameter $\alpha(\hat{x})$ and the gauge potential A are matrix valued. They can be written as: $\alpha = \alpha_r T^r$ and $A = A_r T^r$, where $\alpha_r, A_r \in \mathcal{A}_x$ and the T^r form a suitable basis of matrices. Consider the commutator $[\alpha, A]$, which can be expressed as:

$$[\alpha, A] = \frac{1}{2}(\alpha_r A_s + A_s\alpha_r)[T^r, T^s] + \frac{1}{2}(\alpha_r A_s + A_s\alpha_r)\{T^r, T^s\}. \tag{3.30}$$

If θ^{ij} is real, then the equation (3.1) permits the relation: $(\hat{x}^i)^* = \hat{x}^i$, and one can consider the functions $f(\hat{x})$ to be real: $f^*(\hat{x}) = f(\hat{x})$. This allows α to be hermitian:

$$\alpha(\hat{x}) = \alpha_l(\hat{x})T^l = \alpha^*(\hat{x}), \tag{3.31}$$

where $(\alpha_l(\hat{x}))^\star = \alpha_l(\hat{x})$ and $T_l^\dagger = T_l$. With these considerations, the commutator of $\alpha(\hat{x})$ and $\beta(\hat{y})$ (where $\beta(\hat{y})$ is another hermitian object of the same kind as $\alpha(\hat{x})$) will be antihermitian:

$$([\alpha(\hat{x}), \beta(\hat{y})])^* = -[\alpha(\hat{x}), \beta(\hat{y})]. \tag{3.32}$$

The quantity δA in (3.7) will be hermitian if α, A and $\hat{x}$ are hermitian. It follows that if the matrices T_l form a basis for all hermitian matrices of a certain dimension, then the commutator and anticommutators in (3.30) will also close in these matrices.

Consistency of a noncommutative gauge theory requires the gauge group to be the U(n) group. In [26] it was shown that the Moyal commutator $[X, Y]_\star$ of two functions $X, Y \in$ U(n) closes on U(n). The difficulty with the SO(n), SU(n), and Sp(n) groups is that the Moyal commutator is not necessarily a traceless matrix.

Chapter 4

Some solutions in noncommutative gauge theory in $\mathbb{R}^4_\theta$

4.1 Noncommutative SDYM equations

We consider the set of complex coordinates in $\mathbb{R}^4$ defined as:

$$y = x^1 + \mathrm{i}x^2, \quad z = x^3 - \mathrm{i}x^4, \quad \bar{y} = x^1 - \mathrm{i}x^2, \quad \bar{z} = x^3 + \mathrm{i}x^4, \tag{4.1}$$

and the complex gauge fields:

$$A_y = \frac{1}{2}(A_1 - \mathrm{i}A_2), \quad A_z = \frac{1}{2}(A_3 + \mathrm{i}A_4), \quad A_{\bar{y}} = \frac{1}{2}(A_1 + \mathrm{i}A_2), \quad A_{\bar{z}} = \frac{1}{2}(A_3 - \mathrm{i}A_4). \tag{4.2}$$

With this type of coordinates the SDYM equations (2.12) take the form:

$$F_{yz} = 0, \quad F_{\bar{y}\,\bar{z}} = 0 \quad \text{and} \quad F_{y\bar{y}} + F_{z\bar{z}} = 0. \tag{4.3}$$

Now, we move from ordinary field theory to the noncommutative theory by replacing the function product with the star product:

$$f \star g(x) = \exp\left(\frac{\mathrm{i}}{2}\frac{\partial}{\partial x^i}\theta^{ij}\frac{\partial}{\partial y^j}\right) f(x)g(y)\Big|_{y\to x}.$$

A four-dimensional Euclidean self-dual space is defined by taking the tensor θ^{ij} of the form:

$$\theta^{12} \;=\; -\theta^{21} \;=\; \theta^{34} \;=\; -\theta^{43} \;=\; \theta > 0. \tag{4.4}$$

It will be convenient to work in the operator formulation. Therefore, making use of the Moyal-Weyl transform (3.12) we now work in the noncommutative Euclidean space $\mathbb{R}^4_\theta$ in which the commutation relations for the complex coordinates $y,\ \bar{y},\ z,$ and $\bar{z}$ are:

$$[y, \bar{y}] = 2\theta, \quad [z, \bar{z}] = -2\theta,$$

and the partial derivatives are given by:

$$\partial_y = -\frac{1}{2\theta}[\bar{y},\ ], \quad \partial_z = \frac{1}{2\theta}[\bar{z},\ ],$$

$$\partial_{\bar{y}} = \frac{1}{2\theta}[y,\ ], \quad \partial_{\bar{z}} = -\frac{1}{2\theta}[z,\ ].$$

In the operator formulation the noncommutative version of the SDYM equations (4.3) looks like:

$$-\frac{1}{2\theta}[y, A_z] - \frac{1}{2\theta}[z, A_y] + [A_y, A_z] = 0, \tag{4.5}$$

$$\frac{1}{2\theta}[\bar{y}, A_{\bar{z}}] + \frac{1}{2\theta}[\bar{z}, A_{\bar{y}}] + [A_y, A_z] = 0, \tag{4.6}$$

$$-\frac{1}{2\theta}[\bar{y}, A_{\bar{y}}] - \frac{1}{2\theta}[y, A_y] + [A_y, A_{\bar{y}}] + \frac{1}{2\theta}[\bar{z}, A_{\bar{z}}] + \frac{1}{2\theta}[z, A_z] + [A_z, A_{\bar{z}}] = 0. \tag{4.7}$$

4.2 Reduction of the noncommutative SDYM equations to the difference Nahm equations

In a way analogous to the reduction of the SDYM equations to Nahm's equations exposed in section 2.3, the noncommutative SDYM equations can be reduced to the difference Nahm equations when the ansatz

$$\begin{aligned} A_y &= -2\mathrm{i}\bar{z}T_y(\tau) + 2\mathrm{i}\bar{y}T_{\bar{z}}(\tau), \quad & A_{\bar{y}} &= -2\mathrm{i}\hat{T}_y(\tau)z + 2\mathrm{i}\hat{T}_{\bar{z}}(\tau)y, \\ A_z &= -2\mathrm{i}\bar{y}T_{\bar{y}}(\tau) - 2\mathrm{i}\bar{z}T_z(\tau), \quad & A_{\bar{z}} &= -2\mathrm{i}\hat{T}_{\bar{y}}(\tau)y - 2\mathrm{i}\hat{T}_z(\tau)z, \end{aligned} \tag{4.8}$$

for complex gauge potentials is substituted into (4.5)-(4.7). Here $T_\mu(\tau)$ and $\hat{T}_\mu(\tau)$ are matrix valued functions of $\tau = r^2 = \bar{y}y + \bar{z}z$. The relations between $T_\mu(\tau)$ and $\hat{T}_\mu(\tau)$ will be determined later. The noncommutative product between any function $T_\mu(\tau)$ or $\hat{T}_\mu(\tau)$ and the coordinates, has the following properties:

$$\begin{aligned} zT_\mu(\tau+2\theta) &= T_\mu(\tau)z, \quad \bar{z}T_\mu(\tau) = T_\mu(\tau+2\theta)\bar{z},\\ \bar{y}T_\mu(\tau+2\theta) &= T_\mu(\tau)\bar{y}, \quad yT_\mu(\tau) = T_\mu(\tau+2\theta)y. \end{aligned}$$

When the ansatz (4.8) is inserted into the noncommutative SDYM equations the following equations for T_μ and $\hat{T}_\mu$ are obtained:

From $F_{yz} = 0$:

$$\frac{1}{2\theta}\Big\{T_{\bar{y}}(\tau+2\theta) - T_{\bar{y}}(\tau)\Big\} = -2\mathrm{i}\Big\{T_{\bar{z}}(\tau+2\theta)T_{\bar{y}}(\tau) - T_{\bar{y}}(\tau+2\theta)T_{\bar{z}}(\tau)\Big\}, \tag{4.9}$$

$$\frac{1}{2\theta}\Big\{T_{y}(\tau+2\theta) - T_{y}(\tau)\Big\} = -2\mathrm{i}\Big\{T_{y}(\tau)T_{z}(\tau+2\theta) - T_{z}(\tau)T_{y}(\tau+2\theta)\Big\}, \tag{4.10}$$

$$\begin{aligned} &\frac{1}{2\theta}\Big\{-T_z(\tau+2\theta) + T_z(\tau) + T_{\bar{z}}(\tau-2\theta) - T_{\bar{z}}(\tau)\Big\} = \\ &2\mathrm{i}\Big\{T_{\bar{z}}(\tau-2\theta)T_z(\tau) - T_z(\tau+2\theta)T_{\bar{z}}(\tau) + T_{\bar{y}}(\tau-2\theta)T_y(\tau) - T_y(\tau+2\theta)T_{\bar{y}}(\tau)\Big\}. \end{aligned} \tag{4.11}$$

From $F_{\bar{y}\bar{z}} = 0$:

$$\frac{1}{2\theta}\Big\{\hat{T}_{\bar{y}}(\tau+2\theta) - \hat{T}_{\bar{y}}(\tau)\Big\} = -2\mathrm{i}\Big\{\hat{T}_{\bar{z}}(\tau)\hat{T}_{\bar{y}}(\tau+2\theta) - \hat{T}_{\bar{y}}(\tau)\hat{T}_{\bar{z}}(\tau+2\theta)\Big\}, \tag{4.12}$$

$$\frac{1}{2\theta}\Big\{\hat{T}_{y}(\tau+2\theta) - \hat{T}_{y}(\tau)\Big\} = -2\mathrm{i}\Big\{\hat{T}_{y}(\tau+2\theta)\hat{T}_{z}(\tau) - \hat{T}_{z}(\tau+2\theta)\hat{T}_{y}(\tau)\Big\}, \tag{4.13}$$

$$\begin{aligned} &\frac{1}{2\theta}\Big\{\hat{T}_z(\tau+2\theta) - \hat{T}_z(\tau) + \hat{T}_{\bar{z}}(\tau) - \hat{T}_{\bar{z}}(\tau-2\theta)\Big\} = \\ &2\mathrm{i}\Big\{\hat{T}_y(\tau)\hat{T}_{\bar{y}}(\tau-2\theta) - \hat{T}_{\bar{y}}(\tau)\hat{T}_y(\tau+2\theta) - \hat{T}_{\bar{z}}(\tau)\hat{T}_z(\tau+2\theta) + \hat{T}_z(\tau)\hat{T}_{\bar{z}}(\tau-2\theta)\Big\}. \end{aligned} \tag{4.14}$$

From $F_{y\bar{y}} + F_{z\bar{z}} = 0$:

$$\frac{1}{2\theta}\Big\{\hat{T}_{\bar{z}}(\tau) - \hat{T}_{\bar{z}}(\tau-2\theta) - T_{\bar{z}}(\tau) - T_{\bar{z}}(\tau-2\theta)\Big\} = \\ -2\mathrm{i}\Big\{T_{\bar{z}}(\tau-2\theta)\hat{T}_{\bar{z}}(\tau-2\theta) - \hat{T}_{\bar{z}}(\tau)T_{\bar{z}}(\tau) + T_{\bar{y}}(\tau-2\theta)\hat{T}_{\bar{y}}(\tau-2\theta) - \hat{T}_{\bar{y}}(\tau)T_{\bar{y}}(\tau)\Big\}, \tag{4.15}$$

$$\frac{1}{2\theta}\Big\{-\hat{T}_z(\tau+2\theta) + \hat{T}_z(\tau) + T_z(\tau+2\theta) - T_z(\tau)\Big\} = \\ -2\mathrm{i}\Big\{T_y(\tau+2\theta)\hat{T}_y(\tau+2\theta) - \hat{T}_y(\tau)T_y(\tau) + T_z(\tau+2\theta)\hat{T}_z(\tau+2\theta) - \hat{T}_z(\tau)T_z(\tau)\Big\}, \tag{4.16}$$

$$\frac{1}{2\theta}\Big\{\hat{T}_y(\tau+2\theta) - \hat{T}_y(\tau+4\theta) - T_{\bar{y}}(\tau) + T_{\bar{y}}(\tau+2\theta)\Big\} = \\ -2\mathrm{i}\Big\{-T_{\bar{z}}(\tau+2\theta)\hat{T}_y(\tau+2\theta) + \hat{T}_y(\tau+4\theta)T_{\bar{z}}(\tau) + T_{\bar{y}}(\tau+2\theta)\hat{T}_z(\tau+2\theta) - \hat{T}_z(\tau+4\theta)T_{\bar{y}}(\tau)\Big\}, \tag{4.17}$$

$$\frac{1}{2\theta}\Big\{T_y(\tau+2\theta) - T_y(\tau) - \hat{T}_{\bar{y}}(\tau) + \hat{T}_{\bar{y}}(\tau-2\theta)\Big\} = \\ -2\mathrm{i}\Big\{-T_y(\tau)\hat{T}_{\bar{z}}(\tau) + \hat{T}_{\bar{z}}(\tau-2\theta)T_y(\tau+2\theta) + T_z(\tau)\hat{T}_{\bar{y}}(\tau) - \hat{T}_{\bar{y}}(\tau-2\theta)T_z(\tau+2\theta)\Big\}, \tag{4.18}$$

$$\frac{1}{2\theta}\Big\{\hat{T}_{\bar{z}}(\tau) - T_{\bar{z}}(\tau) - \hat{T}_z(\tau) + T_z(\tau)\Big\} = \\ -2\mathrm{i}\Big\{\hat{T}_y(\tau)T_y(\tau) - \hat{T}_{\bar{z}}(\tau)T_{\bar{z}}(\tau) - \hat{T}_{\bar{y}}(\tau)T_{\bar{y}}(\tau) + \hat{T}_z(\tau)T_z(\tau)\Big\}. \tag{4.19}$$

Imposing the following restrictions between the functions T and $\hat{T}$:

$$\begin{aligned} &\hat{T}_y(\tau+2\theta) = -T_{\bar{y}}(\tau), \quad \hat{T}_z(\tau+2\theta) = -T_{\bar{z}}(\tau), \\ &\hat{T}_{\bar{y}}(\tau) = -T_y(\tau+2\theta), \quad \hat{T}_{\bar{z}}(\tau) = -T_z(\tau+2\theta), \end{aligned} \tag{4.20}$$

and making the necessary shifts to the parameter τ reduces the 11 equations (4.9)-(4.19) to the difference Nahm equations:

$$\frac{1}{2\theta}\Big\{T_{\bar{y}}(\tau+2\theta) - T_{\bar{y}}(\tau)\Big\} = -2\mathrm{i}\Big\{T_{\bar{z}}(\tau+2\theta)T_{\bar{y}}(\tau) - T_{\bar{y}}(\tau+2\theta)T_{\bar{z}}(\tau)\Big\}, \tag{4.21}$$

$$\frac{1}{2\theta}\Big\{T_y(\tau+2\theta)-T_y(\tau)\Big\} = -2\mathrm{i}\Big\{T_y(\tau)T_z(\tau+2\theta)-T_z(\tau)T_y(\tau+2\theta)\Big\}, \tag{4.22}$$

$$\frac{1}{2\theta}\Big\{-T_z(\tau+2\theta)+T_z(\tau)+T_{\bar z}(\tau-2\theta)-T_{\bar z}(\tau)\Big\} = \tag{4.23}$$
$$2\mathrm{i}\Big\{T_{\bar z}(\tau-2\theta)T_z(\tau)-T_z(\tau+2\theta)T_{\bar z}(\tau)+T_{\bar y}(\tau-2\theta)T_y(\tau)-T_y(\tau+2\theta)T_{\bar y}(\tau)\Big\}.$$

From these equations one can recover the commutative version (2.10). In the limit when $\theta \to 0$ these equations take the form:

$$\begin{aligned}\dot T_{\bar y} &= 2\mathrm{i}[T_{\bar z}, T_{\bar y}]\\ \dot T_y &= 2\mathrm{i}[T_y, T_z]\\ \dot T_z + \dot T_{\bar z} &= 2\mathrm{i}\Big\{[T_{\bar z}, T_z]+[T_{\bar y}, T_y]\Big\}.\end{aligned} \tag{4.24}$$

If one applies the change of variables

$$T_y = \frac{1}{2}(T_1 - \mathrm{i}T_2),\;\; T_z = \frac{1}{2}(T_3+\mathrm{i}T_4),\;\; T_{\bar y} = \frac{1}{2}(T_1+\mathrm{i}T_2),\;\; T_{\bar z} = \frac{1}{2}(T_3 - \mathrm{i}T_4),$$

then one obtains Nahm's equations (2.10) in the ordinary commutative $\mathbb{R}^4$.

4.3 Complexified noncommutative one-instanton type solution

A solution to the equations (4.9)-(4.19) and consequently also a solution to the noncommutative SDYM equations (4.5)-(4.7) can be found employing the following ansatz:

$$T_y = \frac{a_y}{\tau+b_y}J_y,\quad T_z = \frac{a_z}{\tau+b_z}J_z,\quad T_{\bar y} = \frac{a_{\bar y}}{\tau+b_{\bar y}}J_{\bar y},\quad T_{\bar z} = \frac{a_{\bar z}}{\tau+b_{\bar z}}J_{\bar z}, \tag{4.25}$$

where $a_y,\ a_z,\ a_{\bar y},\ a_{\bar z},\ b_y,\ b_z,\ b_{\bar y}$ and $b_{\bar z}$ are constants to be determined, and the matrices J are the generators of U(2) of the form:

$$\begin{aligned}J_y &= \tfrac{1}{2}(J_1 - \mathrm{i}J_2) = \tfrac{1}{2\mathrm{i}}\begin{pmatrix}0&0\\1&0\end{pmatrix}, \qquad J_{\bar y} = \tfrac{1}{2}(J_1+\mathrm{i}J_2) = \tfrac{1}{2\mathrm{i}}\begin{pmatrix}0&1\\0&0\end{pmatrix},\\ J_z &= \tfrac{1}{2}(J_3+\mathrm{i}J_4) = -\tfrac{1}{2\mathrm{i}}\begin{pmatrix}0&0\\0&1\end{pmatrix}, \qquad J_{\bar z} = \tfrac{1}{2}(J_3 - \mathrm{i}J_4) = \tfrac{1}{2\mathrm{i}}\begin{pmatrix}1&0\\0&0\end{pmatrix},\end{aligned} \tag{4.26}$$

where

$$J_a = \frac{\sigma_a}{2\mathrm{i}}, \quad J_4 = \frac{\mathbf{1}_2}{2}, \quad a = 1, 2, 3. \tag{4.27}$$

The generators (4.26) have the following properties:

$$\begin{aligned} J_{\bar{y}} J_{\bar{z}} = J_y J_z = 0, \quad J_{\bar{z}} J_{\bar{y}} = \frac{1}{2\mathrm{i}} J_{\bar{y}}, \quad J_z J_y = -\frac{1}{2\mathrm{i}} J_y, \\ J_y J_{\bar{y}} = -\frac{1}{2\mathrm{i}} J_z, \quad J_{\bar{y}} J_y = \frac{1}{2\mathrm{i}} J_{\bar{z}}, \quad J_z J_{\bar{z}} = J_{\bar{z}} J_z = 0. \end{aligned} \tag{4.28}$$

Substituting the ansatz (4.25) in (4.21) and using the properties (4.28) yields the following expression:

$$a_{\bar{z}} = \frac{\tau + b_{\bar{z}} + 2\theta}{\tau + b_{\bar{y}} + 2\theta}. \tag{4.29}$$

If we choose

$$b_{\bar{z}} = b_{\bar{y}} \ \text{ then } \ a_{\bar{z}} = 1. \tag{4.30}$$

In a similar way, from the equation (4.22) the relation:

$$a_z = \frac{\tau + b_z}{\tau + b_y}, \tag{4.31}$$

is obtained, and if we choose

$$b_z = b_y \ \text{ then } \ a_z = 1. \tag{4.32}$$

Inserting the ansatz into the equation (4.23) and using (4.28), we obtain the conditions:

$$a_y a_{\bar{y}} = \frac{\tau + b_{\bar{y}}}{\tau + b_z}, \ \text{ and } \ a_y a_{\bar{y}} = \frac{\tau + b_y}{\tau + b_{\bar{z}}}. \tag{4.33}$$

These conditions will be satisfied if

$$b_z = b_{\bar{y}}, b_y = b_{\bar{z}} \ \text{ and } \ a_y a_{\bar{y}} = 1. \tag{4.34}$$

Finally, from (4.30), (4.32) and (4.34) the following relations between $a_y, a_z, a_{\bar{y}}, a_{\bar{z}}, b_y, b_z, b_{\bar{y}}$ and $b_{\bar{z}}$ are obtained:

$$a_y = 1/a_{\bar{y}} = \alpha, \quad a_z = a_{\bar{z}} = 1, \quad b_z = b_y = b_{\bar{y}} = b_{\bar{z}} = \beta, \tag{4.35}$$

where α and β are real numbers different from zero. Therefore a solution to the difference Nahm equations is given by:

$$T_y = \frac{\alpha}{\tau+\beta} J_y, \quad T_{\bar{y}} = \frac{1}{\alpha(\tau+\beta)} J_{\bar{y}}, \quad T_z = \frac{1}{\tau+\beta} J_z, \quad T_{\bar{z}} = \frac{1}{\tau+\beta} J_{\bar{z}}. \tag{4.36}$$

Employing the form of the solutions (4.36) the relations (4.20) can be expressed as:

$$\begin{aligned}
\hat{T}_{\bar{y}}(\tau) &= -T_y(\tau+2\theta) = -\frac{\alpha}{\tau+\beta+2\theta} J_y, \\
\hat{T}_{\bar{z}}(\tau) &= -T_z(\tau+2\theta) = -\frac{1}{\tau+\beta+2\theta} J_z, \\
\hat{T}_y(\tau) &= -T_{\bar{y}}(\tau-2\theta) = -\frac{1}{\alpha(\tau+\beta-2\theta)} J_{\bar{y}}, \\
\hat{T}_z(\tau) &= -T_{\bar{z}}(\tau-2\theta) = -\frac{1}{\tau+\beta-2\theta} J_{\bar{z}}.
\end{aligned} \tag{4.37}$$

Finally, a complex solution to the noncommutative SDYM equations has the form:

$$\begin{aligned}
A_y &= -2\mathrm{i}\bar{z}T_y(\tau) + 2\mathrm{i}\bar{y}T_{\bar{z}}(\tau), \quad A_{\bar{y}} = -2\mathrm{i}\hat{T}_y(\tau)z + 2\mathrm{i}\hat{T}_{\bar{z}}(\tau)y, \\
A_z &= -2\mathrm{i}\bar{y}T_{\bar{y}}(\tau) - 2\mathrm{i}\bar{z}T_z(\tau), \quad A_{\bar{z}} = -2\mathrm{i}\hat{T}_{\bar{y}}(\tau)y - 2\mathrm{i}\hat{T}_z(\tau)z,
\end{aligned} \tag{4.38}$$

where T_μ and $\hat{T}_\mu$ are defined by (4.36) and (4.37).

4.4 Noncommutative toron type solutions

To solve the equations (4.9)-(4.19) defined in $\mathbb{R}^4_\theta$, we consider the following ansatz for matrix valued functions T_μ and $\hat{T}_\mu$:

$$T_y = T_{\bar{y}} = \hat{T}_y = \hat{T}_{\bar{y}} = 0, \quad \text{and} \quad T_z,\ T_{\bar{z}},\ \hat{T}_z,\ \hat{T}_{\bar{z}} \quad \text{are constant matrices.} \tag{4.39}$$

It is easy to check that for the matrices from (4.39) the conditions (4.20) are satisfied, and therefore, the equations (4.9)-(4.19) reduce to the difference Nahm equations (4.21)-(4.23). Moreover, the conditions (4.20) become

$$\hat{T}_z = T_z^\dagger = -T_{\bar{z}}, \quad \hat{T}_{\bar{z}} = T_{\bar{z}}^\dagger = -T_z, \tag{4.40}$$

and the difference Nahm equations (4.21)-(4.23) reduce to the equation

$$[T_{\bar{z}}, T_z] = 0. \tag{4.41}$$

The ansatz (4.8) for gauge potentials with restrictions (4.39) and (4.40) for matrices $T_z,\ T_{\bar{z}}$ takes the following form:

$$A_y = 2\mathrm{i}\bar{y}T_{\bar{z}}, \quad A_z = -2\mathrm{i}\bar{z}T_z, \quad A_{\bar{y}} = -2\mathrm{i}T_z y, \quad A_{\bar{z}} = 2\mathrm{i}T_{\bar{z}}z. \tag{4.42}$$

The Yang-Mills fields for the gauge potentials (4.42) have the form:

$$\begin{aligned} F_{yz} &= 4\bar{y}\bar{z}[T_{\bar{z}}, T_z], \quad F_{\bar{y}\bar{z}} = 4yz[T_z, T_{\bar{z}}], \\ F_{y\bar{y}} &= 2\mathrm{i}(-T_z - T_{\bar{z}}) - 4\bar{y}y[T_z, T_{\bar{z}}] - 8\theta T_z T_{\bar{z}}, \\ F_{z\bar{z}} &= 2\mathrm{i}(T_z + T_{\bar{z}}) + 4\bar{z}z[T_z, T_{\bar{z}}] + 8\theta T_{\bar{z}} T_z. \end{aligned} \tag{4.43}$$

Imposing (4.41) gives the self-dual Yang-Mills fields

$$\begin{gathered} F_{yz} = 0, \qquad F_{\bar{y}\bar{z}} = 0, \\ F_{y\bar{y}} = -2i(T_z + T_{\bar{z}}) - 8\theta T_z T_{\bar{z}}, \quad F_{z\bar{z}} = 2i(T_z + T_{\bar{z}}) + 8\theta T_{\bar{z}} T_z. \end{gathered} \tag{4.44}$$

So, to each solution of the equation (4.41) we may attribute self-dual gauge potentials (4.42) and self-dual Yang-Mills fields (4.44) in $\mathbb{R}^4_\theta$.

As an example let us choose the following solution of equation (4.41):

$$T_z = aJ_z\ ,\ T_{\bar{z}} = aJ_{\bar{z}}\ ,\ a \in \mathbb{R}\ , \tag{4.45}$$

where J_z and $J_{\bar{z}}$ are generators of the group U(2). The gauge potentials (4.42) with matrices T_z and $T_{\bar{z}}$ from (4.45) are real abelian solutions to the noncommutative generalization of the SDYM equations for the case of self-dual parameter θ. Self-dual abelian Yang-Mills fields have the form

$$F_{yz} = 0, \quad F_{\bar{y}\bar{z}} = 0, \quad F_{y\bar{y}} = -2a\mathrm{i}(J_z + J_{\bar{z}}), \quad F_{z\bar{z}} = 2a\mathrm{i}(J_z + J_{\bar{z}}), \tag{4.46}$$

and they may be considered as noncommutative analogues of the 't Hooft toron solution [27].

Chapter 5

SDYM fields in noncommutative $\mathbb{R}^{2,2}_\theta$ space

5.1 Noncommutative generalization of the CFtHW ansatz in $\mathbb{R}^4_\theta$

The noncommutative generalization of the Corrigan-Fairlie-'t Hooft-Wilczek (CFtHW) ansatz for SDYM fields in $\mathbb{R}^4_\theta$ was introduced in [16]. This ansatz can be modified to obtain an analogous one for the SDYM fields in noncommutative $\mathbb{R}^{2,2}_\theta$.

The noncommutative generalization of the CFtHW ansatz for the gauge potentials in $\mathbb{R}^4_\theta$ takes the form:

$$A_\mu = \bar{\eta}^a_{\mu\nu}\frac{\sigma_a}{2\mathrm{i}}(\phi^{\frac{1}{2}}\partial_\nu\phi^{-\frac{1}{2}} - \phi^{-\frac{1}{2}}\partial_\nu\phi^{\frac{1}{2}}) + \frac{\mathbf{1}_2}{2}(\phi^{-\frac{1}{2}}\partial_\mu\phi^{\frac{1}{2}} + \phi^{\frac{1}{2}}\partial_\mu\phi^{-\frac{1}{2}}). \tag{5.1}$$

The anti-self-dual 't Hooft tensor $\bar{\eta}^a_{\mu\nu}$ is defined in (2.23). When (5.1) is rewritten in terms of the complex coordinates (4.1), it looks:

$$A_{\bar{y}} = \begin{pmatrix} \phi^{-\frac{1}{2}}\partial_{\bar{y}}\phi^{\frac{1}{2}} & -\phi^{-\frac{1}{2}}(\partial_z\phi)\phi^{-\frac{1}{2}} \\ 0 & \phi^{\frac{1}{2}}\partial_{\bar{y}}\phi^{-\frac{1}{2}} \end{pmatrix}, \quad A_{\bar{z}} = \begin{pmatrix} \phi^{-\frac{1}{2}}\partial_{\bar{z}}\phi^{\frac{1}{2}} & \phi^{-\frac{1}{2}}(\partial_y\phi)\phi^{-\frac{1}{2}} \\ 0 & \phi^{\frac{1}{2}}\partial_{\bar{z}}\phi^{-\frac{1}{2}} \end{pmatrix},$$
$$A_y = \begin{pmatrix} \phi^{\frac{1}{2}}\partial_y\phi^{-\frac{1}{2}} & 0 \\ \phi^{-\frac{1}{2}}(\partial_{\bar{z}}\phi)\phi^{-\frac{1}{2}} & \phi^{-\frac{1}{2}}\partial_y\phi^{\frac{1}{2}} \end{pmatrix}, \quad A_z = \begin{pmatrix} \phi^{\frac{1}{2}}\partial_z\phi^{-\frac{1}{2}} & 0 \\ -\phi^{-\frac{1}{2}}(\partial_{\bar{y}}\phi)\phi^{-\frac{1}{2}} & \phi^{-\frac{1}{2}}\partial_z\phi^{\frac{1}{2}} \end{pmatrix}. \tag{5.2}$$

If this ansatz is inserted into equations (4.5)-(4.7) the following equations result:

$$F_{yz} = \begin{pmatrix} 0 & 0 \\ -X & 0 \end{pmatrix} = 0\ , \quad F_{\bar{y}\bar{z}} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} = 0\ , \quad F_{y\bar{y}} + F_{z\bar{z}} = \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix} = 0\ , \tag{5.3}$$

where

$$X = \phi^{-\frac{1}{2}}(\partial_y \partial_{\bar{y}} \phi + \partial_z \partial_{\bar{z}} \phi)\phi^{-\frac{1}{2}}. \tag{5.4}$$

In this way, the noncommutative SDYM equations in $\mathbb{R}^4_\theta$ are reduced to the equation:

$$\phi^{-\frac{1}{2}}(\partial_y \partial_{\bar{y}} \phi + \partial_z \partial_{\bar{z}} \phi)\phi^{-\frac{1}{2}} = 0. \tag{5.5}$$

5.2 SDYM equations in 2+2 dimensions

The self-dual (anti-self-dual) Yang Mills equations in the space $\mathbb{R}^{2,2}$ with metric $(g_{\mu\nu}) = \mathrm{diag}(+1, +1, -1, -1)$ are:

$$\pm F_{\mu\nu} = \tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}. \tag{5.6}$$

Now, let the complex coordinates $y,\ \bar{y},\ z$ and $\bar{z}$ be defined as

$$y = x^1 + \mathrm{i}x^2, \quad z = x^3 + \mathrm{i}x^4, \quad \bar{y} = x^1 - \mathrm{i}x^2, \quad z = x^3 - \mathrm{i}x^4. \tag{5.7}$$

The complex gauge potentials now look:

$$A_y = \frac{1}{2}(A_1 - \mathrm{i}A_2),\ A_z = \frac{1}{2}(A_3 - \mathrm{i}A_4),\ A_{\bar{y}} = \frac{1}{2}(A_1 + \mathrm{i}A_2),\ A_{\bar{z}} = \frac{1}{2}(A_3 + \mathrm{i}A_4). \tag{5.8}$$

The SDYM equations (5.6) expressed in terms of the coordinates (5.7) are [28]:

$$F_{yz} = 0, \quad F_{\bar{y}\,\bar{z}} = 0 \quad \text{and} \quad F_{y\bar{y}} - F_{z\bar{z}} = 0. \tag{5.9}$$

In the noncommutative space $\mathbb{R}^{2,2}_\theta$ the commutation relations between the coordinates (5.7) may be taken in the form:

$$[y, \bar{y}] = 2\theta_1 \quad [z, \bar{z}] = -2\theta_2.$$

Then, the partial derivatives with respect to $y,\ \bar{y},\ z$ and $\bar{z}$ are given by:

$$\partial_y = -\frac{1}{2\theta_1}[\bar{y},\ ],\quad \partial_z = \frac{1}{2\theta_2}[\bar{z},\ ],$$
$$\partial_{\bar{y}} = \frac{1}{2\theta_1}[y,\ ],\quad \partial_{\bar{z}} = -\frac{1}{2\theta_2}[z,\ ].$$

In the operator formulation the noncommutative version of of the SDYM equations in $\mathbb{R}^{2,2}_\theta$ has the same form as (5.9) and we shall refer to them as '(5.9) in $\mathbb{R}^{2,2}_\theta$'.

5.3 Noncommutative generalization of the CFtHW ansatz in $\mathbb{R}^{2,2}_\theta$

The ansatz (5.1) for the gauge potentials A_μ can be adapted for the space $\mathbb{R}^{2,2}_\theta$ in the following way:

$$A_\mu = \bar{f}^a_{\mu\nu}\frac{\sigma_a}{2\mathrm{i}}(\phi^{\frac{1}{2}}\partial^\nu\phi^{-\frac{1}{2}} - \phi^{-\frac{1}{2}}\partial^\nu\phi^{\frac{1}{2}}) + \frac{\mathbf{1}_2}{2}(\phi^{-\frac{1}{2}}\partial_\mu\phi^{\frac{1}{2}} + \phi^{\frac{1}{2}}\partial_\mu\phi^{-\frac{1}{2}}) \tag{5.10}$$

Here $\bar{f}^a_{\mu\nu}$ is a tensor analogous to the anti-self-dual 't Hooft tensor $\bar{\eta}^a_{\mu\nu}$. The tensor $\bar{f}^a_{\mu\nu}$ is defined as [29]:

$$\bar{f}^a_{\mu\nu} = -\bar{f}^a_{\nu\mu} = \begin{cases} f^a_{bc} & \text{with } b=\mu,\ c=\nu, \ \text{ for } \ \mu,\nu = 1,2,3 \\ -\delta^a_\mu & \text{for } \ \nu = 4. \end{cases} \tag{5.11}$$

Here the f^a_{bc} are the structure constants of SU$(1,1)$: $f^1_{23} = f^2_{31} = f^3_{21} = -1$. As in the previous section, the ansatz (5.10) can be written in complex coordinates:

$$A_y = \begin{pmatrix} \phi^{\frac{1}{2}}\partial_y\phi^{-\frac{1}{2}} & 0 \\ \phi^{-\frac{1}{2}}(\partial_{\bar{z}}\phi)\phi^{-\frac{1}{2}} & \phi^{-\frac{1}{2}}\partial_y\phi^{\frac{1}{2}} \end{pmatrix},\ A_z = \begin{pmatrix} \phi^{\frac{1}{2}}\partial_z\phi^{-\frac{1}{2}} & 0 \\ \phi^{-\frac{1}{2}}(\partial_{\bar{y}}\phi)\phi^{-\frac{1}{2}} & \phi^{-\frac{1}{2}}\partial_z\phi^{\frac{1}{2}} \end{pmatrix},$$
$$A_{\bar{y}} = \begin{pmatrix} \phi^{-\frac{1}{2}}\partial_{\bar{y}}\phi^{\frac{1}{2}} & -\phi^{-\frac{1}{2}}(\partial_z\phi)\phi^{-\frac{1}{2}} \\ 0 & \phi^{\frac{1}{2}}\partial_{\bar{y}}\phi^{-\frac{1}{2}} \end{pmatrix},\ A_{\bar{z}} = \begin{pmatrix} \phi^{-\frac{1}{2}}\partial_{\bar{z}}\phi^{\frac{1}{2}} & -\phi^{-\frac{1}{2}}(\partial_y\phi)\phi^{-\frac{1}{2}} \\ 0 & \phi^{\frac{1}{2}}\partial_{\bar{z}}\phi^{-\frac{1}{2}} \end{pmatrix}, \tag{5.12}$$

and then inserted into the equations (5.9) in $\mathbb{R}^{2,2}_\theta$, resulting:

$$F_{yz} = \begin{pmatrix} 0 & 0 \\ -X & 0 \end{pmatrix} = 0\ ,\ F_{\bar{y}\bar{z}} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} = 0\ ,\ F_{y\bar{y}} - F_{z\bar{z}} = \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix} = 0\ , \tag{5.13}$$

with

$$X = \phi^{-\frac{1}{2}}(\partial_y \partial_{\bar{y}} \phi - \partial_z \partial_{\bar{z}} \phi)\phi^{-\frac{1}{2}}. \tag{5.14}$$

Then, a solution to the equation (5.9) in $\mathbb{R}_\theta^{2,2}$ will be found when the following equation for ϕ is solved:

$$\phi^{-\frac{1}{2}}(\partial_y \partial_{\bar{y}} \phi - \partial_z \partial_{\bar{z}} \phi)\phi^{-\frac{1}{2}} = 0. \tag{5.15}$$

5.4 Reduction of the SDYM equations in noncommutative $\mathbb{R}_\theta^{2,2}$ to the sigma model equation in $\mathbb{R}_\theta^{2,0}$

Open $N = 2$ strings in a B-field background induce a noncommutative generalization of a modified U(n) sigma model on the world volume of n coincident $D2$-branes. In the case of n coincident $D1$-branes the standard U(n) sigma model in $1+1$ dimentions and its Euclidean version are obtained [30].

If the following restrictions are applied to the SDYM equation in noncommutative $\mathbb{R}_\theta^{2,2}$:

$$\partial_z A_\mu = \partial_{\bar{z}} A_\mu = 0,$$

then the equations (5.9) in $\mathbb{R}_\theta^{2,2}$ are reduced to the equations for the sigma model in $\mathbb{R}_\theta^{2,0}$:

$$\begin{aligned} \partial_y A_z + [A_y, A_z] &= 0, \\ \partial_{\bar{y}} A_{\bar{z}} + [A_{\bar{y}}, A_{\bar{z}}] &= 0, \\ \partial_y A_{\bar{y}} - \partial_{\bar{y}} A_y + [A_y, A_{\bar{y}}] - [A_z, A_{\bar{z}}] &= 0. \end{aligned} \tag{5.16}$$

We can change the equations (5.16) into the following form:

$$\begin{gathered} \partial_y A_z - \partial_{\bar{y}} A_{\bar{z}} + [A_y, A_z] - [A_{\bar{y}}, A_{\bar{z}}] = 0, \\ \partial_y A_{\bar{y}} - \partial_{\bar{y}} A_y + [A_y, A_{\bar{y}}] - [A_z, A_{\bar{z}}] = 0, \\ \partial_y A_z + \partial_{\bar{y}} A_{\bar{z}} + [A_y, A_z] + [A_{\bar{y}}, A_{\bar{z}}] = 0. \end{gathered} \tag{5.17}$$

If we chose the gauge such that:

$$A_z = A_{\bar{y}}, \;\; A_{\bar{z}} = A_y,$$

then the equations (5.17) become:

$$\partial_y A_{\bar{y}} - \partial_{\bar{y}} A_y + 2[A_y, A_{\bar{y}}] = 0, \tag{5.18}$$

$$\partial_y A_{\bar{y}} + \partial_{\bar{y}} A_y = 0. \tag{5.19}$$

The following form of the gauge potentials solve the equation (5.18):

$$A_y = \frac{1}{2}\Phi^{-1}\partial_y \Phi, \;\; A_{\bar{y}} = \frac{1}{2}\Phi^{-1}\partial_{\bar{y}}\Phi, \tag{5.20}$$

where $\Phi(y,\bar{y})$ is a group-valued function, while substituting (5.20) in (5.19) leads to the sigma model equation in noncommutative $\mathbb{R}^{2,0}_\theta$:

$$\partial_y(\Phi^{-1}\partial_{\bar{y}}\Phi) + \partial_{\bar{y}}(\Phi^{-1}\partial_y\Phi) = 0. \tag{5.21}$$

5.5 Solution of the sigma model equations in noncommutative $\mathbb{R}^{2,0}_\theta$

If the restrictions $\partial_z\phi = \partial_{\bar{z}}\phi = 0$ are applied to the matrices (5.12) then we obtain:

$$A_y = \begin{pmatrix} \phi^{\frac{1}{2}}\partial_y\phi^{-\frac{1}{2}} & 0 \\ 0 & \phi^{-\frac{1}{2}}\partial_y\phi^{\frac{1}{2}} \end{pmatrix}, \; A_z = \begin{pmatrix} 0 & 0 \\ \phi^{-\frac{1}{2}}(\partial_{\bar{y}}\phi)\phi^{-\frac{1}{2}} & 0 \end{pmatrix},$$

$$A_{\bar{y}} = \begin{pmatrix} \phi^{-\frac{1}{2}}\partial_{\bar{y}}\phi^{\frac{1}{2}} & 0 \\ 0 & \phi^{\frac{1}{2}}\partial_{\bar{y}}\phi^{-\frac{1}{2}} \end{pmatrix}, \; A_{\bar{z}} = \begin{pmatrix} 0 & -\phi^{-\frac{1}{2}}(\partial_y\phi)\phi^{-\frac{1}{2}} \\ 0 & 0 \end{pmatrix}. \tag{5.22}$$

When these matrices are inserted into the noncommutative sigma model equations (5.16) they take the form:

$$\begin{pmatrix} 0 & 0 \\ -X & 0 \end{pmatrix} = 0\,, \quad \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} = 0\,, \quad \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix} = 0, \tag{5.23}$$

with

$$X = \phi^{-\frac{1}{2}}(\partial_y\partial_{\bar{y}}\phi)\phi^{-\frac{1}{2}}. \tag{5.24}$$

In this way, a solution to equation (5.16) will be obtained when a function ϕ satisfying the equation

$$\phi^{-\frac{1}{2}}(\partial_y \partial_{\bar{y}} \phi)\phi^{-\frac{1}{2}} = 0, \tag{5.25}$$

is found. For instance, a function of the form:

$$\phi(y, \bar{y}) = f(y) + g(\bar{y}),$$

will satisfy the equation (5.25).

Conclusions and results

In analogy with the commutative case (as exposed in [22]), we succeeded to reduce the noncommutative SDYM equations to the difference Nahm equations in $\mathbb{R}^4_\theta$ by using a special form for the gauge potentials.

Having constructed solutions to the difference Nahm equations, we have obtained two types of solutions to the noncommutative SDYM equations in $\mathbb{R}^4_\theta$. The first one is a noncommutative 1-instanton type complex solution. The other noncommutative solution corresponds to a toron type solution, where the gauge potentials linearly depend on complex coordinates.

Considering the noncommutative generalization of the CFtHW ansatz for the SDYM equations in $\mathbb{R}^{2,2}_\theta$ [16], and making the gauge potentials independent of two coordinates, we have written down a solution to the sigma model equations in $\mathbb{R}^{2,0}_\theta$, (which corresponds to the effective field theory on the Euclidean $D1$-brane world volume [30]).

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I am very grateful to Prof. Dr. Olaf Lechtenfeld for giving me the opportunity to make this thesis work at the Institut für Theoretische Physik, Universität Hannover. I also owe much to Dr. Alexander D. Popov, for his direction of this work, as well as to Dr. Tatiana Ivanova for her guide during the realization of this work.
I am indebted to A. Elizabeth Nuncio-Quiroz and Jens Könemann for the revision of the text and for their suggestions.
I am also grateful to my parents, my brother and my sister. To Prof. Dr. Martin Henzler and to the Albert-Ludwig-Fraas-Stiftung, to Mrs. Maria Esther Quiroz Amarillas and Mr. Elpidio Nuncio Rodriguez, to Hélène Zielasek, Fátima Sulzbacher and Hans Sulzbacher.

Specially I want to thank my mother for her support during my studies.

Hiermit erkläre ich, die Diplomarbeit selbstständig verfasst und keine anderen als die angegebenen Hilfsmittel verwendet zu haben.

Hannover, 28. Mar., 2002