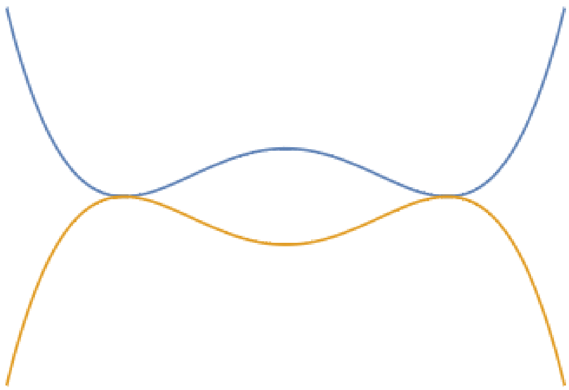
Coset space dimensional reduction of Yang–Mills on non-compact symmetric spaces

Master Thesis



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20.02.2023
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Ehrenwörtliche Erklärung

Hiermit versichere ich, die vorliegende Abschlussarbeit selbständig und nur unter Verwendung der von mir angegebenen Quellen und Hilfsmittel verfasst zu haben. Sowohl inhaltlich als auch wörtlich entnommene Inhalte wurden als solche kenntlich gemacht. Die Arbeit hat in dieser oder vergleichbarer Form noch keinem anderen Prüfungsgremium vorgelegen.

Datum: 20.02.2023 Unterschrift: Mahir Ertürk

Gutachter: Prof. Dr. Olaf Lechtenfeld
Zweitgutachter: Prof. Dr. Domenico Giulini

Acknowledgements

First and foremost I want to express my gratitude towards my supervisor Prof. Dr. Olaf Lechtenfeld for taking me on as his masters student and part of his group in the institute. Working under his supervision, I was treated as an equal and felt included; I could always rely on his help during my studies and was provided resources to make my endeavor as enjoyable as possible. I am especially thankful for him going out of his way to advice, guide and support me through all the aspects necessary to further follow the path of becoming an academic researcher. Thanks to him I had the opportunity to learn what it is like to be part of a research group, give talks and write papers, which not only was very fun but also an immense help.

In the same vein I want to thank my second examiner Prof. Dr. Domenico Giulini for co-examining my thesis. Beside his useful comments on my thesis, he also greatly advises and supports me during my first steps in laying food in academia, for which I am extremely grateful.

Next up I want to thank the other members in Prof. Lechtenfelds group. Most notably I thank Gabriel Picanço Costa, Kaushlendra Kumar and Savan Hirpara for useful discussions and generally having a fun time doing physics and discussing all sorts of topics surrounding physics and academia.

I also want to thank my fellow students M. von Boehn, J. Hertzberg and A. Müller. Knowing each other since the beginning of the bachelors, we have endured all the hardships and also the fun of our studies together and have become close friends in the process. I certainly would not have made it to this point without them.

Finally, I want to thank my family, most notably my mother and my uncle. All throughout my life they have supported me in all aspects of my existence, be they emotional, creative or financial. Since I was little, they have always encouraged me to follow my passions and have gone out of their way to enable me with the possibilities to do so. Without them, there is no doubt, I would not have had the chance to follow this deepest passion of mine – physics. I am infinitely grateful for having them by my side.

Contents

1 Introduction and summary

The Yang–Mills equations in full generality are difficult to solve by virtue of being a system of non-linear partial differential equations. It is hence imperative to require further conditions on either the underlying spacetime, the gauge field itself or both to make the problem more approachable. In particular requiring the gauge field to have certain symmetries with respect to the spacetime coordinates can help one to reduce the degrees of freedom substantially, thus yielding much more solvable equations for a subclass of symmetric solutions. Such approaches are usually called 'dimensional reduction'. A particular incarnation of this is the 'coset space dimensional reduction' (CSDR) scheme [1]. There one generally considers spacetimes which partly consist of a coset space G/H, thus making it possible to require the gauge field to be invariant under the natural G action on spacetime. A special case of this scenario is Yang–Mills theory with gauge group G over spacetime which is just a cylinder over a coset $M = \mathbb{R} \times G/H$. In this case one even obtains that the Yang–Mills equations reduce to a system of ordinary non-linear (matrix-)equations.

Such setups were considered in particular by my supervisor and collaborators over the years, e.g. [2], [3], [4] and more recently [5]. Additionally they considered scenarios where the spacetimes where cylinders over Lie groups, e.g. [6], [7], [8], [9], in which subclasses of symmetric solutions can be naturally mapped to solutions of corresponding CSDR setups. In many of these cases the Yang–Mills system reduces to one or two Newton-like degrees of freedom $\phi(t)$ subject to some quartic potential, hence yielding analytic and symmetric solutions to the Yang–Mills equations once solved. One can exploit this approach even more by introducing warping functions into the cylinders [2] or gluing different cylinders, on which the symmetrized equations coincide, together [5].

In particular in [2] a warping function was used to obtain solutions on de Sitter space dS_n via the spherical slicing $\mathbb{R} \times S^{n-1}$. There the spheres were written as three different types of cosets, one of them being $SO(n)/SO(n-1)$, which even is a symmetric space. In that case, when dropping the effects of the warping, they found that the system reduces to a single Newton-like degree of freedom $\phi(t)$ subject to a double well potential. The main goal of this thesis is to investigate the CSDR scheme over cylinders with gauge groups $SO(1,n)$ and $SO(2,n-1)$ for the cosets hyperbolic space $H^n \cong SO(1,n)/SO(n)$, de Sitter space $dS_n \cong SO(1,n)/SO(1,n-1)$ and anti-de Sitter space $AdS_n \cong SO(2,n-1)/SO(1,n-1)$, all of which in these representations are non-compact symmetric spaces.

The thesis is structured into two main parts. The first part, consisting of chapter two and three, is a quick but still extensive introduction into the mathematical apparatus of differential geometry, general relativity and gauge theory. We cover many of the most important takeaways and core concepts of courses in these fields including calculus on manifolds, (pseudo-)Riemannian geometry and the geometry of Lie groups and homogeneous spaces, as well as bundle theory, principal bundles and Yang–Mills theory. The second part of the thesis is the main endeavor consisting of chapter four to six. With the mathematical groundwork laid up, in chapter four we will introduce the coset space dimensional reduction scheme over cylinders. Chapter five deals with the application of CSDR to the three non-compact symmetric spaces at hand. We find that in all cases considered the system reduces to a single Newton-like degree of freedom, which is subject to either a double well or inverted double well potential. Even more so, we find that the reduced Lagrangians of all the systems, including the case $S^n \cong SO(n+1)/SO(n)$ in [2], are the same except of the sign of the quartic potential. Thereafter we solve the equations of motion analytically, as the solutions of a particle subject to a double well are given by Jacobi elliptic functions. We also derive a closed expression for the energy-momentum tensors for all the cases, which in case of the Riemannian slicings with S^n and H^n naturally are of perfect fluid structure. Despite of the presence of non-compact gauge groups, all energy densities are finite for bounded solutions and (can be made) positive, though the actions and over all energies naturally diverge. In chapter six we first consider the four dimensional case $\mathbb{R} \times H^3$ which can naturally be coupled to gravity with the corresponding FLRW type open hyperbolic ansatz. This is in analogy to the case of $\mathbb{R} \times S^3$ with $S^3 \cong SU(2)$ considered in [10], [8], though the setup there is not CSDR but the aforementioned construction with cylinders over groups (and not cosets). The conformal invariance of Yang–Mills in four spacetime dimensions yields a partial decoupling of the full Einstein–Yang–Mills equations. Working in conformal time, the Friedmann equations for the scale factor then also become that of a Newton-like particle subject to a quartic potential whose, behavior is determined by the sign of the cosmological constant; The coupling to YM is then solely through an energy balancing condition for the mechanical 'energies' of the analog Newtonian particles. In the second section of chapter six we consider the influence of a general warping function on the equations of motion. We find that the equations of motion change by the addition of a Hubble friction term which of course drops out in four spacetime dimensions. Finally, in third and last section of chapter six, we consider the hyperbolic slicing of anti-de Sitter space which can be obtained by warping the hyperbolic CSDR setup. This is also in analogy to [2] where the same thing was considered with the spherical slicing of de Sitter space. We find that the Hubble friction term is not as well behaved away from spacetime dimension three as it becomes non-dissipative, which is the exact opposite as to what happens in the de Sitter case. For spacetime dimension three we numerically estimate the region of initial conditions in phase space, for which the solutions will stay bounded, which becomes non trivial due to the presence of the friction term.

2 Riemannian geometry

In this chapter we briefly review key concepts of differential geometry. Despite keeping the discussion at the bare essentials, we still develop all the objects and structures from the ground up and focus on the most important takeaways of a rather broad course on differential geometry. We begin with the calculus of 'bare' manifolds, that is, manifolds without any further structure imposed. In this we especially cover the notions of tensors, differential forms, integration and the different kinds of mappings between manifolds. After having established these essentials we move on to (pseudo-)Riemannian geometry in the second part. There we cover core concepts of manifolds endowed with a metric, including connections, covariant derivatives and the different notions of curvature. The third and fourth part then focus on Lie groups, Lie algebras and homogeneous spaces. Finally in the fifth part we give a brief introduction into General Relativity, covering only necessary definitions and results which we will need for the main endeavor of the thesis. The aspects discussed in this chapter are all standard material and are a compilation of [11], [12], [13], [14], [15], [16], [17].

2.1 Calculus on manifolds

Definition 2.1. A (real) **manifold** of dimension d is a Hausdorff topological space M which

(i) fulfills the second axiom of countability

(ii) is locally euclidian, that is, $\forall m \in M\ \exists$ open neighbourhood $U \subset M$and a homeomorphism $\phi : U \to \Omega \subset \mathbb{R}^d$.

Furthermore we call a pair (U, ϕ) **local chart** or **local coordinates** and a set $\mathcal{A} = \{(U_i, \phi_i)\}_{i\in I}$ an **atlas** of M if the charts fully cover M.
For two charts with overlapping regions $U_{ij} = U_i \cap U_j \neq \emptyset$ we call the map $\phi_j \circ \phi_i^{-1} : \phi_i(U_{ij}) \to \phi_j(U_{ij})$ **chart transition map** or **coordinate transformation**. Also, since the chart transition maps map between subsets of $\mathbb{R}^d$, we have a notion of differentiability. Thus we call an atlas whose transition maps are all of type $\mathcal{C}^k$ a $\mathcal{C}^k$ atlas. A manifold admitting a $\mathcal{C}^k$ atlas is called $\mathcal{C}^k$ manifold. Manifolds that are $\mathcal{C}^\infty$ are called smooth and we will from now on restrict all discussions to smooth manifolds if not stated otherwise. Finally, a manifold which is a subset of another manifold is called a **submanifold**.

Theorem 2.2. *Let $F : \mathbb{R}^m \to \mathbb{R}^n$ be differentiable. If $p \in \mathbb{R}^n$ is a regular value, that is, $DF|_{F^{-1}(p)}$ is surjective, the set $M := F^{-1}(p)$ is a submanifold of $\mathbb{R}^m$ with dimension $\dim(M) = m - n$.*

Example 2.3.

(i) $\mathbb{R}^d$ is a smooth d dimensional manifold with a global chart given by the identity.

(ii) The n-Sphere $S^n := \{\vec{x} = (x_1, ..., x_{n+1}) \mid \|\vec{x}\|^2 = 1\} \subset \mathbb{R}^{n+1}$ is a n-dimensional submanifold of $\mathbb{R}^{n+1}$.

The essential idea of manifolds is that they are spaces, which locally look like ordinary euclidian space $\mathbb{R}^d$. That is, we can describe the spaces purely by looking at them through local charts without losing any information. Of course this idea and also the terminology is alluding to actual charts as we know them from our day to day world. The remarkable thing is that this formalism allows us to simply ignore 'where' the space lives. That is, we do not necessarily have to talk about manifolds as submanifolds of some other bigger space. Even more so, every object abstractly defined on a manifold can also be *fully* characterized by its representation in local charts. This in turn enables us, in the same fashion as for the chart transition maps, define a notion of differentiation for all sorts of objects that live on manifolds by simply looking at their local representation in a chart. This leads us to the following definition.

Definition 2.4. Let M be a manifold and $f : M \to \mathbb{R}$ a continuous map. We then call f $\mathcal{C}^k$ if $f \circ \phi : \Omega \subset \mathbb{R}^d \to \mathbb{R}$ is $\mathcal{C}^k$ for every local chart ϕ. We denote by the set of these functions with $\mathcal{C}^k(M)$.

Before proceeding we introduce a convention. From now on we will refer to the charts $\phi : U \subset M \to \Omega \subset \mathbb{R}^n$ as $x(p) \in \Omega$ and denote the inverse $x^{-1} : \Omega \to U$ with $\phi = x^{-1}$. We do this to make it easier for the eye to understand that the components of the chart $x^i : U \to \mathbb{R}$ (later on also called x^μ) are really the coordinates.

Definition 2.5. Let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a smooth curve with $\gamma(0) = p$ and $\dot{\gamma}(0) \equiv \frac{\mathrm{d}}{\mathrm{d}t}\gamma|_{t=0} =: v$ in some chart. Furthermore introduce the equivalence relation for two such curves via

$$\gamma \sim \delta :\Leftrightarrow \dot{\gamma}(0) = \dot{\delta}(0) \quad \text{in some chart.} \tag{2.1}$$

We then call

$$T_pM := \{\gamma : (-\varepsilon, \varepsilon) \to M \mid \gamma \text{ smooth} \ , \ \gamma(0) = p\}/\sim \tag{2.2}$$

the **tangent space** of M at the point p. An element of T_pM we call **tangent vector** of M at the point p. Each tangent space at every point indeed is a vector space with the same dimension as the underlying manifold.

Definition 2.6. Let M be a smooth manifold, we then call the disjoint union of all tangent spaces

$$TM := \bigcup_{p \in M} T_pM \tag{2.3}$$

the **tangent bundle**[1] of M. Furthermore we call a smooth map

$$X : M \to TM \tag{2.4}$$
$$p \mapsto X|_p \in T_pM \tag{2.5}$$

a **vector field** on M. We denote the set of all smooth vector fields on a manifold by $\mathfrak{X}(M)$. Via point wise scalar multiplication, the space of vector fields $\mathfrak{X}(M)$ becomes a $\mathcal{C}^\infty(M)$ module with the same dimension as the underlying manifold.

Tangent vectors and vector fields allow us to define directional derivatives on manifolds.

Definition 2.7. Let M be a smooth manifold, $f \in \mathcal{C}^\infty(M)$ and $X \in \mathfrak{X}(M)$. We then define the action of X on f via

$$X|_p(f) := \frac{\mathrm{d}}{\mathrm{d}t} f \circ \gamma\Big|_{t=0} \tag{2.6}$$

where γ is a representative of $X|_p \in T_pM$. Doing this for every point, we define the **differential** of f as

$$f \mapsto \mathrm{d}f(X) := X(f) \in \mathcal{C}^\infty(M). \tag{2.7}$$

Indeed, every vector field $X \in \mathfrak{X}(M)$ is a derivation, meaning they act linearly and obey the Leibniz rule

$$(gX + Y)(f) = gX(f) + Y(f) \tag{2.8}$$
$$X(f\,g) = X(f)\,g + f\,X(g) \ \ \forall f, g \in \mathcal{C}^\infty(M). \tag{2.9}$$

Since everything behaves like point wise linear algebra, we can expand vector fields in bases. Given a local chart (x, U) on M, there exists a special basis of vector fields called *coordinate basis*. We denote these as the partial derivatives with respect to the coordinates x^i, precisely because they act like so on functions. They are defined via their action as

$$\frac{\partial}{\partial x^i}\Big|_p(f) \equiv \partial_i|_p(f) := \frac{\mathrm{d}}{\mathrm{d}t} f\left(x^{-1}(x(p) + t\,\vec{e}_i)\right)\Big|_{t=0}. \tag{2.10}$$

More generally, given a basis $\{E_i\} \subset \mathfrak{X}(M)$, any vector field can be expanded as

$$X|_p = X^i(p)\,E_i|_p \ , \ \ X^i(p) \in \mathcal{C}^\infty(M). \tag{2.11}$$

After having introduced the tangent spaces, we now consider their dual spaces.

[1] Chapter 3 will go deeper into the bundle formalism.

Definition 2.8. Let M be a smooth manifold and $p \in M$. We then call the dual space of the tangent space at p

$$T_p^*M := (T_pM)^* = \{\omega : T_pM \to \mathbb{R} \mid \text{ linear } \} \tag{2.12}$$

the **cotangent space** at p. As usual it is the set of linear maps from T_pM into $\mathbb{R}$. The union of all cotangent spaces

$$T^*M := \bigcup_{p\in M} T_p^*M \tag{2.13}$$

is called the **cotangent bundle**. A smooth map

$$\omega : M \to T^*M \tag{2.14}$$
$$p \mapsto \omega|_p \in T_p^*M \tag{2.15}$$

is called **one-form** on M. We denote the set of all one-forms on a manifold by $\Omega^1(M)$.

Naturally, the space of one-forms also forms a $\mathcal{C}^\infty(M)$ module with the same dimension as the underlying manifold. One-forms map vector fields into scalar functions in a linear fashion. Even more so, due to the point wise linearity, they are $\mathcal{C}^\infty(M)$ linear in their argument.

$$(f\,\omega + \theta)(X) = f\,\omega(X) + \theta(X) \tag{2.16}$$
$$\omega(f\,X) = f\,\omega(X) \;\; \forall X \in \mathfrak{X}(M)\,, f \in \mathcal{C}^\infty(M) \tag{2.17}$$

A special case of one-forms, are the differentials of functions. For $f \in C^\infty(M)$ we have

$$\mathrm{d}f \in \Omega^1(M). \tag{2.18}$$

Because of this, we will from now on call the space of smooth functions $\Omega^0(M) := \mathcal{C}^\infty(M)$. We can thus see that the differential

$$\mathrm{d} : \Omega^0(M) \to \Omega^1(M) \tag{2.19}$$

is a derivation on $\Omega^0(M)$. Like for the vector fields, the one-forms also have a *coordinate basis*, when a local chart (x, U) is given. The basis is given by the differentials of the coordinate functions $x^i : U \to \mathbb{R}$. We have

$$\mathrm{d}x^i|_p(X) := X(x^i)|_p. \tag{2.20}$$

In particular we have

$$\mathrm{d}x^i|_p\left(\frac{\partial}{\partial x^j}\right) = \frac{\mathrm{d}}{\mathrm{d}t}x^i\left(x^{-1}(x(p) + t\,\vec{e}_j)\right)|_{t=0} = \delta^i{}_j. \tag{2.21}$$

Any two bases $\{E_i\} \subset \mathfrak{X}(M)$ and $\{e^j\} \subset \Omega^1(M)$ with this property

$$e^i(E_j) = \delta^i{}_j \tag{2.22}$$

are called *dual* to each other. Hence, coordinate bases are always dual. In this case, applying any of the basis one-forms to a vector field expanded in the dual, just projects out the corresponding component function. Furthermore the differential of a function $f \in \Omega^0(M)$ can then be expanded as

$$\mathrm{d}f = E_i(f)\,e^i. \tag{2.23}$$

Changing a basis by applying a point wise linear transformation $T^i{}_j(p) \in \mathrm{End}(T_pM)$, i.e. $E_i \mapsto E'_i = T^j{}_i E_j$, yields the familiar transformation behavior for the components of vector fields and one-forms respectively

$$X^i \mapsto (T^{-1})^i{}_j X^j = X'^i \tag{2.24}$$
$$\omega_i \mapsto T^j{}_i \omega_j = \omega'_i. \tag{2.25}$$

That is, vector components transform with the inverse transpose of T and form components with T itself. We call these transformation behaviors *contra- and covariant* respectively and they are automatically captured by the respective index positions up and down. In particular we have for a coordinate transformation $x \mapsto x'(x)$ that the transformation is just the Jacobian.

From the tangent- and cotangent bundles we can generalize to the tensor bundles.

Definition 2.9. Let M be a smooth manifold and $p \in M$. We then consider the tensor product of the r-fold tensor product of T_pM with itself and the s-fold tensor product of T_p^*M with it self.

$$T_p^{(r,s)}M := \underbrace{T_pM \otimes ...T_pM}_{r} \otimes \underbrace{T_p^*M \otimes ...T_p^*M}_{s} \tag{2.26}$$

Then, the disjoint union

$$T^{(r,s)}M := \bigcup_{p \in M} T_p^{r,s}M \tag{2.27}$$

is called **tensor bundle of rank** (r,s). A smooth map

$$T : M \to T^{(r,s)}M \tag{2.28}$$

$$p \mapsto T|_p \in T_p^{(r,s)}M \tag{2.29}$$

is called (rank (r,s)-)**tensor (field)** on M. We denote the set of all (r,s)-tensor fields simply by $\Gamma(T^{(r,s)}M)$.

Since tensor fields are composite objects of vector fields and one-forms, we already know all of their properties by virtue of multiliniearity. Indeed, everything behaves like point wise linear algebra. Given bases $\{E_i\} \subset \mathfrak{X}(M)$ and $\{e^j\} \subset \Omega^1(M)$ any tensor can be locally expanded as

$$T = T^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} E_{i_1} \otimes ... \otimes E_{i_r} \otimes e^{j_1} \otimes ... \otimes e^{j_s}. \tag{2.30}$$

Under basis change, every index transforms in multilinear fashion co- and contravariantly as caputred by the index notation.

Definition 2.10. Let M be a smooth n-dimensional manifold, $p \in M$ and $k \in [1,n] \subset \mathbb{N}$. We then consider the the k-fold exterior power of the cotangent space T_p^*M.

$$\Lambda_p^k(M) := \Lambda^k(T_p^*M) \tag{2.31}$$

Then, the disjoint union

$$\Lambda^k(T^*M) := \bigcup_{p \in M} \Lambda^k(T_p^*M) \tag{2.32}$$

is called k**-form bundle**. A smooth map

$$\omega : M \to \Lambda^k(T^*M) \tag{2.33}$$

$$\omega \mapsto \omega|_p \in \Lambda_p^k(M) \tag{2.34}$$

is called k**-form** on M. We denote the set of all k-forms by $\Omega^k(M)$.

Of course the set of k-forms is a subset of the $(0,k)$-tensor fields $\Omega^k(M) \subset \Gamma(T^{(0,k)}M)$. A k-form is thus a smooth collection of totally anti-symmetric, multi-$\mathcal{C}^\infty$-linear mappings from $\mathfrak{X}(M)^k$ to $\mathbb{R}$.

$$\omega(X_1,...,X_k) = \mathrm{sgn}(\sigma)\omega(X_{\sigma(1)},...,X_{\sigma(k)})\ , \quad X_1,...X_k \in \mathfrak{X}(M)\ ,\ \omega \in \Omega^k(M) \tag{2.35}$$

In a local basis we can expand a k-form as

$$\omega = \frac{1}{k!}\sum_\sigma \mathrm{sgn}(\sigma)\,\omega_{i_1,\ldots,i_k}\, e^{\sigma(i_1)} \otimes ... \otimes e^{\sigma(i_k)} \tag{2.36}$$

$$=: \omega_{i_1,\ldots,i_k}\, e^{[i_1} \otimes ... \otimes e^{i_k]} \tag{2.37}$$

$$=: \frac{1}{k!}\omega_{i_1,\ldots,i_k}\, e^{i_1} \wedge ... \wedge e^{i_k} \tag{2.38}$$

where we have introduced the familiar wedge product in the algebra of totally anti-symmetric tensors[2]. The dimension of $\Omega^k(M)$ as $\mathcal{C}^\infty$ module is $\binom{n}{k}$, where n is the dimension of the underlying manifold M.

After having introduced the basics of (co-)tangent structures on manifolds, we now look at the types of maps between two manifolds.

[2]Also called Graßmann-Alebgra.

Definition 2.11. Let M, N be two smooth manifolds of dimensions m, n with charts (U_M, x), (U_N, y) respectively. Furthermore let $\varphi : M \to N$ be a smooth map. φ then induces maps between the cotangent- and tangent bundle of M and N respectively.

(i) The **pushforward** of φ

$$\varphi_* : \mathfrak{X}(M) \to \mathfrak{X}(N) \tag{2.39}$$
$$V \mapsto \varphi_* V \tag{2.40}$$

is then defined via

$$\varphi_* V(f) := V(f \circ \varphi)\ , \quad f \in \Omega^0(N). \tag{2.41}$$

In a local chart it is represented by the differential of $\varphi^k \equiv y^k \circ \varphi \circ x^{-1}$

$$\varphi_*(V^i \frac{\partial}{\partial x^i}) = V^i D\varphi(\frac{\partial}{\partial x^i}) = V^i \frac{\partial \varphi^k}{\partial x^i} \frac{\partial}{\partial y^k}. \tag{2.42}$$

In particular this means that we have a point wise linear mapping between $T_pM \to T_{\varphi(p)}N$.

(ii) The **pullback** of φ

$$\varphi^* : \Omega^1(M) \leftarrow \Omega^1(N) \tag{2.43}$$
$$\omega \mapsto \varphi^* \omega \tag{2.44}$$

is defined via

$$\varphi^* \omega(V) := \omega(\varphi_* V)\ , \quad V \in \mathfrak{X}(M). \tag{2.45}$$

In a local chart we have by virute of $\mathcal{C}^\infty$-linearity

$$\varphi^*(\omega_i \mathrm{d}y^i) = \omega_i \frac{\partial \varphi^i}{\partial x^k} \mathrm{d}x^k. \tag{2.46}$$

In particular this means that we have a point wise linear mapping between $T_p^*M \leftarrow T_{\varphi(p)}^*N$.

Both operations generalize to arbitrary totally contra- and covariant tensor fields respectively by virtue of multilinearity. In case that φ is a diffeomorphism, we can define the pull back of vector fields and push forward of forms by pushing and pulling like above but with φ^{-1}. In this case the operations generalize to tensor fields of arbitrary rank.

Note that the last remark in the definition, that is for diffeomorphisms, implies the familiar transformation behavior of tensors via the inverse transpose of the Jacobian for contravariant indicies and via the Jacobian for covariant indicies.

Having established pushforwards and pullbacks, we can now classify different types of maps between manifolds.

Definition 2.12. Let M, N be two smooth manifolds of dimension m, n respectively. Furthermore let $\varphi : M \to N$ be a smooth map. We then call φ

(i) **submersion** at $p \in M$ if in a neighborhood φ_* is (point wise) surjective

(ii) **immersion** at $p \in M$ if in a neighborhood φ_* is (point wise) injective

We call these attributes **global** if they hold for all $p \in M$. Furthermore we call $\varphi : M \to \varphi(M) \subset N$ an **embedding** if it is an immersion and a homeomorphism when restricted to its image. We remark that if φ is a diffeomorphism, φ_* is point wise bijective.

The intuition behind these terms is straight forward. A submersion is a map that, in general, squishes a big manifold M into a smaller one N ($m \geq n$) with self intersections. An immersion, in general, puts a smaller manifold M into a bigger one N ($m \leq n$) also with self intersections. An embedding puts a smaller manifold M into a bigger one without self intersections. A prime example for the latter are submanifolds of $\mathbb{R}^d$, which are all embedded (by definition). Working with embedded manifolds (especially in $\mathbb{R}^d$) is a big topic in and of itself hence there is a big theorem concerning them.

Theorem 2.13 (Whitney)**.** *Let M be a smooth m-dimensional (real) manifold. Then, it can always be smoothly embedded into $\mathbb{R}^{2m}$.*

Finally, we briefly cover the main aspects of calculus on manifolds, that is, differentiation and integration. If no further structure is provided, we mainly consider three types of differentiation; The *exterior-*, *Lie-* and the *covariant derivative.* The latter we will cover in the next chapter.

Definition 2.14. Let $X \in \mathfrak{X}(M)$. A curve $\gamma : I \to M$ is called **integral curve** of X at $p \in M$ if its velocity coincides with X.

$$\dot{\gamma}|_t = X|_{\gamma(t)} \ , \ \ \gamma(0) = p \tag{2.47}$$

We also say that γ flows with X. In coordinates we have

$$\dot{\gamma}|_t = \dot{\gamma}^i(t) \frac{\partial}{\partial x^i}\Big|_{\gamma(t)} = X^i(\gamma(t)) \frac{\partial}{\partial x^i}\Big|_{\gamma(t)} . \tag{2.48}$$

Thus, integral curves are given by solutions to a first order ODE. The existence of (most) integral curves is guaranteed at least locally by the Picard-Lindelöf theorem.

Definition 2.15. Let $X \in \mathfrak{X}(M)$. The **flow** of X is defined as

$$\Theta_X(t) : M \to M \tag{2.49}$$
$$p \mapsto \Theta_X(t) := \gamma_p(t) \tag{2.50}$$

where γ_p is the integral curve of X at p. Hence, for small enough t, $\Theta_X(t)$ is a diffeomorphism of M into itself.

Since every vector field induces a diffeomorphism, we can use it to define a derivative by pushing the fields along the flow.

Definition 2.16. Let $X \in \mathfrak{X}(M)$ and $T \in \Gamma(T^{(r,s)}M)$. Furthermore denote let $\Theta_X(t)$ be the flow of X. We then define the **Lie derivative** of T with respect to X as

$$\mathcal{L}_X T := \frac{\mathrm{d}}{\mathrm{d}t} \Theta_X(t)^* T\Big|_{t=0} \in \Gamma(T^{(r,s)}M). \tag{2.51}$$

In local dual bases $\{E_i\}$, $\{e^j\}$ it is given by

$$(\mathcal{L}_X T)^{i_1 \dots i_r}_{j_1 \dots j_s} = X(T^{i_1 \dots i_r}_{j_1 \dots j_s}) - \sum_\ell E_k(X^{i_\ell}) T^{i_1 \dots i_{\ell-1}\, k\, i_{\ell+1} \dots i_r}_{j_1 \dots j_s} + \sum_\ell E_{j_\ell}(X^k) T^{i_1 \dots i_r}_{j_1 \dots j_{\ell-1}\, k\, j_{\ell+1} \dots j_s} . \tag{2.52}$$

Particularly interesting is the Lie derivative of a vector field, which is characterized by its action on functions, we have

$$(\mathcal{L}_X Y)(f) = X(Y(f)) - Y(X(f)) =: [X, Y](f) \tag{2.53}$$

where we have introduced the *commutator* of vector fields

$$[\cdot, \cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M). \tag{2.54}$$

This *anti-symmetric* product not only turns $\mathfrak{X}(M)$ into an algebra but even a *Lie*-algebra, which has the additional property

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0 \tag{2.55}$$

called the *Jacobi-identity.* Naturally the Lie derivative is a derivation on the space of tensor fields, where the Leibniz rule is understood to hold along the tensor product.

Definition 2.17. Let M be a smooth n-dimensional manifold and $\omega \in \Omega^k(M)$ a k-form. The **exterior derivative**

$$\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M) \tag{2.56}$$

is then defined via

$$\mathrm{d}\omega(X_0, ..., X_k) := \sum_{i=0}^{k}(-1)^i X_i(\omega(X_0, ..., \hat{X}_i, ..., X_k) + \sum_{0\leq i\leq j\leq k}(-1)^{i+j}\omega([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_k). \tag{2.57}$$

In a local basis it is given by

$$\mathrm{d}(\omega_{i_1...i_k} e^{i_1} \wedge ... \wedge e^{i_k}) = \mathrm{d}(\omega_{i_1...i_k}) \wedge e^{i_1} \wedge ... \wedge e^{i_k} + \omega_{i_1...i_k}\mathrm{d}(e^{i_1} \wedge ... \wedge e^{i_k}) \tag{2.58}$$

where $\mathrm{d} : \Omega^0(M) \to \Omega^1(M)$ is the differential of functions.

The exterior derivative obeys the following important properties.

(i) $\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^{\deg(\alpha)}\alpha \wedge \mathrm{d}\beta$

(ii) $\mathrm{d} \circ \mathrm{d} \equiv 0$

(iii) $\mathrm{d}(\varphi^*\omega) = \varphi^*(\mathrm{d}\omega)$

Property (i) means that it is an *anti-derivation.* From property (ii) we especially get for a coordinate basis

$$\mathrm{d}(\omega_{i_1...i_k}\mathrm{d}x^{i_1} \wedge ... \wedge \mathrm{d}x^{i_k}) = (\partial_k\omega_{i_1...i_k}))\mathrm{d}x^k \wedge \mathrm{d}x^{i_1} \wedge ... \wedge \mathrm{d}x^{i_k} \tag{2.59}$$

$$= \mathrm{d}(\omega_{i_1...i_k}) \wedge \mathrm{d}x^{i_1} \wedge ... \wedge \mathrm{d}x^{i_k} \tag{2.60}$$

We also introduce new terminology. A form ω whose differential is zero $\mathrm{d}\omega = 0$ we call *closed* and a form ω who is the differential of some other form α, i.e. $\omega = \mathrm{d}\alpha$, we call *exact.* From property (ii) we see that every exact form is also closed. The converse is also true if the underlying domain is a star domain, this fact is known as the *Poincaré Lemma.*

Lastly, we introduce integration on manifolds. Integration on manifolds is closely related to the calculus of differential forms.

Definition 2.18. Let M be a smooth n-dimensional manifold. We then call a form $\omega \in \Omega^n(M)$ **volume form** on M if it is nowhere vanishing, that is, $\omega|_p \neq 0 \ \forall p \in M$. Since $\Omega^n(M)$ is one-dimensional as a $\mathcal{C}^\infty$ module, every n-form and volume form can be expressed locally as

$$\omega = \omega(x)\mathrm{d}x^1 \wedge ... \wedge \mathrm{d}x^n \tag{2.61}$$

Theorem 2.19. *Let M be a smooth manifold. We call M **orientable** if it has an atlas where the Jacobians of all the transition maps have either positive or negative determinant. With this definition we get*

$$M \text{ orientable} \Leftrightarrow \exists \text{ volume form.} \tag{2.62}$$

Definition 2.20. Let M be a smooth manifold and $\{(x_i, U_i)\}$ an atlas. A collection of maps $\sigma_i : U_i \to \mathbb{R}$ is called **partition of unity** if

$$0 \leq \sigma_i \leq 1 \text{ and } \sum_i \sigma_i(p) = 1 \ \forall p \in M \tag{2.63}$$

Together with these definitions we can now define integration.

Definition 2.21. Let M be a smooth, orientable manifold, $\{(x_i, U_i)\}$ an atlas with $\phi_i := x_i^{-1}$, $\sigma_i : U_i \to \mathbb{R}$ a corresponding partition of unity and $\alpha \in \Omega^n(M)$. The **integral** of the form α is then defined as

$$\int_M \alpha := \sum_i \int_{x(U_i)} \phi_i^*(\sigma_i\alpha) \tag{2.64}$$

In coordinates we have

$$\int_{x(U_i)} \phi_i^*(\sigma_i\alpha) = \int_{x(U_i)} \sigma_i(x)\alpha(x)\mathrm{d}x^1 \wedge ... \wedge \mathrm{d}x^n := \int_{x(U_i)} \sigma_i(x)\alpha(x)\mathrm{d}^n x \tag{2.65}$$

where $\sigma_i(x)$ and $\alpha(x)$ are the local representations respectively. Furthermore, for a function $f \in \Omega^0(M)$ and a volume form $\omega \in \Omega^n(M)$ we define the integral of f with respect to ω as

$$\int_M f \mathrm{d}\mu_\omega := \int_M (f\,\omega) \tag{2.66}$$

The definitions can also be localized by simply restricting the integration region.

This formalism entails a strong advantage as it captures the correct transformation under change of coordinates manifestly.

Theorem 2.22. *Let M and N be smooth, orienteble, n-dimensional manifolds, $\omega \in \Omega^n(N)$ and $\Phi : M \to N$ a diffeomorphism. Then*

$$\int_{\Phi(M)=N} \omega = \int_M \Phi^* \omega. \tag{2.67}$$

Locally we have

$$\int_{y(\Phi(M))=y(N)} \omega(y) \mathrm{d}y^1 \wedge ... \wedge \mathrm{d}y^n = \int_{x(M)} \omega(\Phi(x)) \det(D\Phi(x)) \mathrm{d}x^1 \wedge ... \wedge \mathrm{d}x^n. \tag{2.68}$$

The statement also holds for local diffeomorphisms. For the special case that $N = M$ we get the familiar transformation rule for coordinate transformations.

The n-fold wedge product together with the transformation behavior of differential forms leads naturally to the determinant of the Jacobian, which we know from ordinary multivariable calculus. Finally, we introduce a powerful theorem which is the generalization of the fundamental theorem of calculus.

Theorem 2.23 (Stokes)**.** *Let M be a smooth, orientable, n-dimensional manifold, ∂M its boundary and $\omega \in \Omega^{n-1}(M)$. Then*

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega. \tag{2.69}$$

Note that this also holds if the manifold has no boundary, that is, $\partial M = \emptyset$. In that case we have

$$\int_M \mathrm{d}\omega = 0 \text{ for } \partial M = \emptyset. \tag{2.70}$$

2.2 Riemannian metrics and curvature

Up until now we have understood manifolds merely as sets which locally look like Euclidean space. Aside from rough topological properties, the spaces had no sense of geometry, that is, notions of distance, curvature and so on. To analyze these properties, we need to provide some further structure called a *metric*. The study of manifolds with a metric is called (pseudo-)Riemannian geometry and is a powerful and elegant way of doing geometry, which lies at the heart of many physical theories.

Definition 2.24. Let M be a smooth n-dimensional manifold. A tensor field $g \in \Gamma(T^{(0,2)}M)$ is called **metric** on M if for all $p \in M$ $g|_p$ is a symmetric and non-degenerate (bilinear)form on $T_pM \times T_pM$. Furthermore if g is positive definite, we call it **Riemannian metric**, if it is of indefinite signature (p, q) we call it **pseudo-Riemannian metric** and, as a special case, when it is of signature $(1, n-1) \equiv (n-1, 1)$ we call it **Lorentzian metric**. The pair (M, g) is called (pseudo-)Riemannian- and Lorentzian manifold respectively.

Thus a metric is a smooth, point wise distribution of non-degenerate inner products on the tangent spaces. Through this, a metric defines what lengths tangent vectors have and what the angles between them are. Locally a metric can be expressed as

$$g = g_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j \ , \ \ g_{ij} = g_{ji} = g(\partial_i, \partial_j). \tag{2.71}$$

It is common practise to define the components of the inverse of the metric as

$$g^{ij} := (g^{-1})^{ij} \text{ that is } g^{ik} g_{kj} = \delta^i{}_j. \tag{2.72}$$

By point wise diagonalization of the components, which is always possible for symmetric forms, we can always find local dual bases (which are in general not coordinate bases) such that

$$g = g_{ij} e^i \otimes e^j \quad \text{with} \quad g_{ij} = g(E_i, E_j) = \text{diag}(+1, ..., +1, -1, ..., -1) \tag{2.73}$$

we call such bases *orthonormal*. We also introduce the convention that

$$\text{diag}(\underbrace{+1, ..., +1}_{p}, \underbrace{-1, ..., -1}_{q}) =: \eta_{ij}^{(p,q)} \tag{2.74}$$

Definition 2.25. Let (M, g_1) and (N, g_2) be two manifolds with metrics. Furthermore let $M \times N$ be the product manifold together with the natural projections

$$\pi_1 \;: M \times N \to M \tag{2.75}$$
$$\pi_2 \;: M \times N \to N. \tag{2.76}$$

We then call the metric defined on $M \times N$ via

$$g := \pi_1^* g_1 + \pi_2^* g_2 \tag{2.77}$$

the **product metric** of g_1 and g_2. Furthermore let $\varphi : M \to \mathbb{R}_{>0}$ be a smooth, positive function. We then call

$$g := \pi_1^* g_1 + (\pi_1^* \varphi) \pi_2^* g_2 \tag{2.78}$$

warped product metric and the pair $(M \times_\varphi N, g)$ a warped product manifold. Finally, in the special case that $M \cong \mathbb{R}$, $g_1 = \mathrm{d}u \otimes \mathrm{d}u$, we call the warped product, **warped cylinder** over N.

When dealing with a (pseudo-)Riemannian manifold, we change the definition of volume forms from before.

Definition 2.26. Let (M, g) be a n-dimensional, (pseudo-)Riemannian manifold. We then call a form $\mathrm{dVol} \in \Omega^n(M)$ **volume form** on (M, g), if its local representation a chart it is

$$\mathrm{dVol} = \sqrt{|g|}\, \mathrm{d}x^1 \wedge ... \wedge \mathrm{d}x^n \tag{2.79}$$

where $|g| := |\det(g_{ij})|$. This definition is independent of choice of coordinates and in particular, when $\{e^i\}$ is an orthonormal dual frame we have

$$\mathrm{dVol} = e^1 \wedge ... \wedge e^n. \tag{2.80}$$

Further more we define

$$\mathrm{Vol}(M) := \int_M \mathrm{dVol} \tag{2.81}$$

as the **volume** of M.

The introduction of a metric g also introduces more structure on tensor fields, which we will briefly review now.

Definition 2.27. Let (M, g) be a (pseudo-)Riemannian manifold and. We then define the **musical isomorphisms** of tangent and cotangent bundle via

$$\flat : \mathfrak{X}(M) \to \Omega^1(M)\,, \qquad \sharp :\; \Omega^1(M) \to \mathfrak{X}(M) \tag{2.82}$$
$$X \mapsto X^\flat := g(X, \cdot)\,, \qquad \omega \mapsto \omega^\sharp \;\text{ such that }\; g(\omega^\sharp, X) = \omega(X). \tag{2.83}$$

Locally this means

$$(X^\flat)_j = X^i g_{ij}\,, \quad (\omega^\sharp)^j = \omega_i g^{ij}. \tag{2.84}$$

From this we define more generally the **lifting and lowering of indices** of arbitrary tensors as the contraction with the metric (lowering) and with the inverse metric (lifting). Through this we can change the rank of tensors.

Definition 2.28. Let (M, g) be a n-dimensional, (pseudo-)Riemannian manifold and $\{E_i\} \subset \mathfrak{X}(M)$ an orthonormal basis. Furthermore let $\omega_1, \omega_2 \in \Omega^k(M)$. We then define the **inner product of forms** as

$$\langle \omega_1, \omega_2 \rangle := \sum_{1 \leq i_1 < ... < i_k \leq n} \omega_1(E_{i_1}, ..., E_{i_k}) \omega_2(E_{i_1}, ..., E_{i_k}). \tag{2.85}$$

In particular for two one-forms we have

$$\langle \alpha, \beta \rangle =: g(\alpha, \beta) = g(\alpha^\sharp, \beta^\sharp) = g^{ij} \alpha_i \beta_j \equiv \alpha_i \beta^j. \tag{2.86}$$

Definition 2.29. For $\omega_1, \omega_2 \in \Omega^k(M)$ we define the **Hodge star operator** $* : \Omega^k(M) \to \Omega^{n-k}(M)$ via

$$\langle \omega_1, \omega_2 \rangle \, \mathrm{dVol} =: \omega_1 \wedge (*\omega_2). \tag{2.87}$$

The Hodge star operator is a $\mathcal{C}^\infty$-linear isomorphism between the space of k forms and $n-k$ forms. In an orthonormal basis $\{e^i\}$ it is given by

$$*(e^{\sigma(1)} \wedge ... \wedge e^{\sigma(k)}) = e^{\sigma(k+1)} \wedge ... \wedge e^{\sigma(n)}. \tag{2.88}$$

It is also closely related to the volume form via

$$*1 = \mathrm{dVol} \tag{2.89}$$

from which we can define the integrals of functions $f \in \Omega^0(M)$ as

$$\int_M f := \int_M *f = \int_M f \, \mathrm{dVol}. \tag{2.90}$$

Finally it obeys the following properties for $\omega_{1,2} \in \Omega^k(M)$

$$* * \omega = (-1)^{k(n-k)} \omega \tag{2.91}$$

$$\langle *\omega_1, *\omega_2 \rangle = \langle \omega_1, \omega_2 \rangle \,. \tag{2.92}$$

Definition 2.30. Let (M, g), (N, h) be two (pseudo-)Riemannian manifolds and $\varphi : M \to N$ a diffeomorphism. We then call φ an **isometry** if

$$g = \varphi^* h. \tag{2.93}$$

In this case we call the two spaces **isometric**. Furthermore we call φ a **conformal map** if

$$\varphi^* h = f \, g \tag{2.94}$$

for some function $f \in \Omega^0(M)$. In this case we call the spaces **conformally equivalent**.

Given a metric, we can define the lengths of curves on a manifold. We do this in total analogy to the way we would do it in physics.

Definition 2.31. Let (M, g) be a (pseudo-)Riemannian manifold and $\gamma : I \subset \mathbb{R} \to M$ a piecewise smooth curve, that is, we have a partition $I = (I_0, I_1) \cup (I_1, I_2) \cup ... \cup (I_{n-1}, I_n)$ where γ is smooth on each component. We then define the **length** of γ as

$$L(\gamma) := \sum_i \int_{I_i}^{I_{i+1}} \sqrt{g(\dot\gamma(s), \dot\gamma(s))} \mathrm{d}s. \tag{2.95}$$

Note that if g is not Riemannian, L can be less then or equal to zero even if γ is not constant. Finally, we see that the definition is invariant under reparametrisation of the curve.

Definition 2.32. Let (M, g) be a (pseudo-)Riemannian manifold. We call a vector field $X \in \mathfrak{X}(M)$ **Killing vector field** if

$$\mathcal{L}_X g \equiv 0 \tag{2.96}$$

the Lie derivative of the metric in the direction X vanishes.
The space of Killing vector fields is a Lie subalgebra of $\mathfrak{X}(M)$ and one can show that its dimension as a $\mathbb{R}$-vector space has an upper bound

$$\dim(\mathrm{Kill}(M, g)) \leq \frac{1}{2} n(n+1) \tag{2.97}$$

where n is the dimension of M. Furhtermore we call a metric manifold **maximally symmetric** if the dimension of the space of Killing fields attains the upper bound. It is easy to see that the Killing fields are the generators of isometries, that is, their flows are always isometric.

We now introduce the third kind of derivative on manifolds called the *covariant derivative* or equivalently a *connection.* We will see that connections entail the notion of curvature and furthermore that a manifold with metric always has a unique special connection whose curvature is induced by the metric.

Definition 2.33. Let M be a smooth manifold. We then call a map $\nabla : \mathfrak{X}(M) \times \Gamma(T^{(r,s)}M) \to \Gamma(T^{(r,s)}M)$ a **connection** or **covariant derivative** on M if

(i) $\nabla_X f = X(f)$

(ii) $\nabla_{fX} T = f\, \nabla_X T$

(iii) $\nabla_X (T + S) = \nabla_X T + \nabla_X S$

(iv) $\nabla_X (T \otimes S) = \nabla_X T \otimes S + T \otimes \nabla_X S$

(v) $\nabla_X \circ C = C \circ \nabla_X$

where $f \in \Omega^0(M)$, $X \in \mathfrak{X}(M)$, $T, S \in \Gamma(T^{(r,s)}M)$ (rank can be different for (iv)) and where C denotes the contraction. Given a basis $\{E_i\} \subset \mathfrak{X}(M)$ with dual basis $\{e^i\} \subset \Omega^1(M)$ the local representation of covariant derivatives of vector fields is given by

$$\nabla_{E_i} E_j = \Gamma_{i\,j}^{\,k} E_k \tag{2.98}$$

where we have introduced the **Christoffel symbols** or **connection one-forms**

$$\Gamma^k_{\ j} = \Gamma_{i\,j}^{\,k} e^i \in \Omega^1(M). \tag{2.99}$$

That is, for a vector field $X = X^i E_i$ we have

$$\nabla_{E_i} X = E_i(X^k) E_k + \Gamma_{i\,j}^{\,k} X^j E_k \tag{2.100}$$

$$\Rightarrow \quad (\nabla X)^k = \mathrm{d}X^k + \Gamma^k_{\ j} X^j. \tag{2.101}$$

Togetether with the Christoffel symbols, we get the general expression for the covariant derivative of an arbitrary tensor field, we have

$$(\nabla_{E_i} T)^{a_1,\dots,a_r}_{b_1,\dots,b_s} = E_i(T^{a_1\dots a_r}_{b_1\dots b_s}) + \sum_\ell \Gamma_{i\,c}^{\,a_l} T^{a_1\dots a_{\ell-1}\,c\,a_{\ell+1}\dots a_r}_{b_1\dots b_s} - \sum_\ell \Gamma_{i\,b_\ell}^{\,c} T^{a_1\dots a_r}_{b_1\dots b_{\ell-1}\,c\,b_{\ell+1}\dots b_s}. \tag{2.102}$$

The covariant derivative thus acts like the ordinary derivative plus contributions where the Christoffel symbols transform the components linearly. Indeed, the Christoffel symbols can be understood as $\mathrm{End}(TM)$ valued one-forms. Despite of the structure of their indices they should not be confused with tensors. Connection one-forms are their own class of geometric objects, which are defined by their transformation behavior. For a local change of basis $E_i|_p \mapsto \bar{E}_k|_p = J^i_{\ k}(p) E_i|_p$ a connection transforms via

$$\bar{\Gamma}_{i\ j}^{\ k} = (J^{-1})^k_{\ b} \Gamma_{a\ c}^{\ b} J^c_{\ j} J^a_{\ i} + (J^{-1})^k_{\ \ell} \bar{E}_i(J^\ell_{\ j}). \tag{2.103}$$

Hence, the the first part of the transformation is truly that of a tensor with respective rank but the second part is new. In other words we have

$$\bar{\Gamma} = \Gamma + J^{-1}\mathrm{d}J \tag{2.104}$$

where we understand J as a $\mathrm{End}(TM)$ valued function.

We remark that given two different connections $\nabla, \bar{\nabla}$ their difference $\nabla - \bar{\nabla}$ becomes a rank $(1, 2)$ tensor field whose components are given by the difference of the connection one-forms $\Gamma - \bar{\Gamma}$. This means that the space of connections, that is, the space of connection one-forms, is an affine space.

Given a covariant derivative on a manifold, one can define the covariant derivative along a curve. Skipping the nuances of this definition, we can define it heuristically by simply expanding the tangent vector of the curve in a local basis and then restricting the general expression onto the image of the curve. This leads us to define the acceleration of a curve and in particular curves without acceleration.

Definition 2.34. Let ∇ be a connection and $\gamma : I \to M$ a curve. The **acceleration** $\ddot{\gamma} \in \mathfrak{X}(M)\Big|_{\mathrm{Im}(\gamma)}$ of γ is then defined as

$$\ddot{\gamma}|_{\gamma(s)} := \nabla_{\dot{\gamma}}\dot{\gamma}|_{\gamma(s)} = \left(\ddot{\gamma}^k(s) + \Gamma_{i\ j}^{\ k}(\gamma(s))\dot{\gamma}^i\dot{\gamma}^j\right) E_k|_{\gamma(s)}. \tag{2.105}$$

Furthermore a curve γ whose acceleration is identically zero is called a **geodesic** and is given by the solution of the second order, non-linear ODE

$$\ddot{\gamma}^k(s) + \Gamma_{i\ j}^{\ k}(\gamma(s))\dot{\gamma}^i\dot{\gamma}^j = 0 \tag{2.106}$$

called the **geodesic equation**.

Definition 2.35. Let ∇ be a connection. We then define the **torsion** $T \in \Gamma(T^{(1,2)}M)$ of ∇ as

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]. \tag{2.107}$$

Due to the anti-symmetry in its arguments, the torsion can be understood as a vector valued two-form. In a local basis it takes the form

$$T = \frac{1}{2}T^k_{ij}E_k \otimes e^i \wedge e^j \tag{2.108}$$

where

$$T^k_{ij} = \Gamma_{i\ j}^{\ k} - \Gamma_{j\ i}^{\ k} - f^k_{ij} \tag{2.109}$$

and $[E_i, E_j] =: f^k_{ij}E_k$ is the local expression for the Lie bracket. If we choose a coordinate basis $E_i = \partial_i$ the Lie bracket vanishes and we get that

$$T^k_{ij} = \Gamma_{i\ j}^{\ k} - \Gamma_{j\ i}^{\ k}. \tag{2.110}$$

Note that if the Christoffel symbols are symmetric in the lower two indices, the torsion vanishes. When dealing with the dual frame, we can also express the torsion in a compact way as

$$T^k = \mathrm{d}e^k + \Gamma^k_{\ j} \wedge e^j \tag{2.111}$$

or equivalently, when contracting with $\otimes E_k$ we get

$$T = \mathrm{d}e + \Gamma \wedge e \quad \text{First Cartan structure equation} \tag{2.112}$$

where we understand that the Christoffel symbols act on the e^j via matrix multiplication.

Up until now we have not included the presence of a metric in our discussion on connections. It turns out that when a metric is given, there is one special connection which is adapted to it, called the *Levi-Civita connection.* To define it we have to introduce two properties that characterize it uniquely.

Definition 2.36. Let (M,g) be a (pseudo-)Riemannian manifold and ∇ a connection. We then call ∇

(i) **metric**, if $\nabla g \equiv 0$

(ii) **torsion free**, if its torsion vanishes.

Note that torsion free implies that the Christoffel symbols are symmetric in the lower indices

Theorem 2.37. *Let (M,g) a (pseudo-)Riemannian manifold. Then, there is a unique connection ∇ which is both metric and torsion free, called the* ***Levi-Civita*** *connection. Furthermore in this case the* ***Koszul formula*** *holds.*

$$\begin{aligned} 2g(\nabla_X Y, Z) =& X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ &+ g([X,Y],Z) - g([Y,Z],X) - g([X,Z],Y) \end{aligned} \tag{2.113}$$

The Christoffel symbols of the Levi-Civita connection in a coordinate basis ∂_i read

$$\Gamma_{i\ j}^{\ k} = \frac{1}{2}g^{k\ell}(g_{\ell i,j} + g_{\ell j,i} - g_{ij,\ell}) \tag{2.114}$$

where an index after a comma denotes differentiation with respect to that basis field.

In most cases when discussing geometry, one usually talks about the Levi-Civita connection, since in this case the object which provides curvature - the connection - is derived from the object which measures lengths and angels - the metric. Furthermore one can show that the geodesics of the Levi-Civita connection extremise the length functional defined by the metric.[3]

We are now in a place to define the different notions of curvature which are provided by the presence of a connection.

Definition 2.38. Let ∇ be a connection. We then define the **curvature tensor** $R \in \Gamma(T^{(1,3)}M)$ as

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{2.115}$$

Evidently it is anti-symmetric in its first two arguments and hence can be viewed as a $\mathrm{End}(TM)$ valued two form. In a local basis it reads

$$R = \frac{1}{2} R^a{}_{bij} E_a \otimes e^b \otimes e^i \wedge e^j \tag{2.116}$$

and in terms of the Christoffel symbols we have

$$R^a{}_{bij} = E_i(\Gamma_j{}^a{}_b) - E_j(\Gamma_i{}^a{}_b) + \Gamma_i{}^a{}_\ell \Gamma_j{}^\ell{}_b - \Gamma_j{}^a{}_\ell \Gamma_i{}^\ell{}_b - f^\ell{}_{ij} \Gamma_\ell{}^a{}_b \tag{2.117}$$

and for a coordinate basis $E_i = \partial_i$ we have

$$R^a{}_{bij} = \partial_i \Gamma_j{}^a{}_b - \partial_j \Gamma_i{}^a{}_b + \Gamma_i{}^a{}_\ell \Gamma_j{}^\ell{}_b - \Gamma_j{}^a{}_\ell \Gamma_i{}^\ell{}_b. \tag{2.118}$$

We can also write it in a compact form as

$$R = \mathrm{d}\Gamma + \Gamma \wedge \Gamma \quad \text{Second Cartan structure equation} \tag{2.119}$$

where we again understand the products involved as matrix multiplication.

Depending on the connection, the curvature tensor has certain symmetry properties. In general it always is anti-symmetric in the last, lower two entries. If the connection is torsion free we additionally have

$$R^a{}_{bij} + R^a{}_{jbi} + R^a{}_{ijb} = 0 \tag{2.120}$$

and if the connection is metric we have

$$R_{abij} = -R_{baij}. \tag{2.121}$$

Finally, notice that the curvature tensor can be used to compensate the exchange of order when taking two covariant derivatives.

Definition 2.39. Let (M,g) be a (pseudo-)Riemannian manifold, ∇ the Levi-Civita connection and R the corresponding curvature tensor. We then call the totally covariant version of the curvature

$$Riem(W,X,Y,Z) := g(W, R(X,Y)Z) \tag{2.122}$$

the **Riemann tensor**. Hence, we have locally

$$Riem_{abij} = g_{a\ell} R^\ell{}_{bij} =: R_{abij}. \tag{2.123}$$

Since the Riemann tensor comes from (the unique) metric and torsion free connection, it obeys

$$Riem(W,X,Y,Z) = -Riem(W,X,Z,Y) \tag{2.124}$$
$$Riem(W,X,Y,Z) = -Riem(X,W,Y,Z) \tag{2.125}$$
$$Riem(W,X,Y,Z) = Riem(Y,Z,W,X) \tag{2.126}$$
$$Riem(W,X,Y,Z) + Riem(W,Z,X,Y) + Riem(W,Y,X,Z) = 0 \tag{2.127}$$

implying that the number of independent components of $Riem$ is

$$\#Riem = \frac{1}{12} n^2 (n^2 - 1) \tag{2.128}$$

where $n \geq 2$ is the dimension of M.

[3] Indeed it is a common trick to vary the length functional to obtain all the non-zero components of the Christoffel symbols.

Definition 2.40. Let (M, g) be a (pseudo-)Riemannian manifold, ∇ the Levi-Civita connection and R the corresponding curvature tensor. We then define the **sectional curvature** k as

$$k(X, Y) := \frac{Riem(X, Y, X, Y)}{g(X, X)g(Y, Y) - g(X, Y)^2}. \tag{2.129}$$

This definition only depends on the the span of $X|_p, Y|_p$. The sectional curvature determines all components of the Riemann tensor. Furthermore, we say that a manifold has **constant sectional curvature** if $k|_p$ does not depend on its arguments, i.e. the sectional curvature does not depend on the direction.

Remark 2.41. If (M, g) is a manifold with constant sectional curvature $k|_p$, its Riemann tensor can be expressed as

$$Riem = \frac{1}{2} kg \owedge g \tag{2.130}$$

where $\owedge$ denotes the Kulkarni-Nomizu product of two symmetric rank (0,2) tensors

$$(T \owedge S)_{abij} := T_{ai}S_{bj} + T_{bj}S_{ai} - T_{aj}S_{bi} - T_{bi}S_{aj}. \tag{2.131}$$

Theorem 2.42 (Gauss' Theorema Egregium)**.** *Let (M, g) be a two-dimensional, Riemannian manifold. Then, the sectional curvature $k|_p$ is the Gaussian curvature, proving that curvature is an intrinsic property, i.e. it only depends on g.*

Definition 2.43. Let ∇ be a connection and R the corresponding curvature tensor. We then define the **Ricci tensor** $Ric \in \Gamma(T^{(0,2)}M)$ as

$$Ric_{ij} := R^k_{\ ikj} =: R_{ij} \tag{2.132}$$

the partial trace of the Riemann tensor. That is

$$Ric(X, Y) = \mathrm{tr}(Z \mapsto R(Z, X)Y). \tag{2.133}$$

In case of the Levi-Civita connection we have

$$R_{ij} = g^{\ell k} R_{\ell ikj} \tag{2.134}$$

implying that Ric is symmetric.

Definition 2.44. Let (M, g) be a (pseudo-)Riemannian manifold, ∇ the Levi-Civita connection and R the corresponding curvature. We then define the **Ricci scalar** $\mathcal{R}$ as

$$\mathcal{R} := \mathrm{tr}_g Ric = g^{ij} R_{ij} \tag{2.135}$$

the trace of the Ricci tensor. The Riemann scalar is related to the sectional curvatures via

$$\mathcal{R} = \sum_{i,j} k(E_i, E_j) \tag{2.136}$$

where $\{E_i\}$ is an orthonormal frame.

When working with connections and curvature it is often cumbersome to calculate all the quantities of interest. Luckily there are strong, coordinate free expressions relating all our quantities yielding shortcuts for calculations. Two such important relations are known as the *Bianchi identities*.

Theorem 2.45. *Let M be a smooth manifold, ∇ a connection, Γ, T, R the Christoffel symbols, torsion and curvature and $\{E_i\}, \{e^j\}$ a dual frame. Then, the* ***Bianchi identites*** *hold.*

$$\mathrm{d}T + \Gamma \wedge T = R \wedge e \tag{2.137}$$

$$\mathrm{d}R + \Gamma \wedge R - R \wedge \Gamma = 0 \tag{2.138}$$

where we understand all the products to be matrix multiplication.

Definition 2.46. A (pseudo-)Riemannian manifold is called **Einstein**, if $Ric = k\,g$ for some $k \in \mathbb{R}$.

Theorem 2.47 (Schurr's Lemma)**.** *Let (M, g) be a (pesudo-)Riemannian manifold of dimension $n \neq 2$. If the Ricci tensor has the form*

$$Ric|_p = k|_p\, g|_p \tag{2.139}$$

for some function $k \in \Omega^0(M)$, then

$$\mathrm{d}k \equiv 0. \tag{2.140}$$

In particular we have for spaces of constant sectional curvature $k|_p$, that k is constant and that

$$Ric = (n-1)k\, g \tag{2.141}$$
$$\mathcal{R} = n(n-1)k. \tag{2.142}$$

Hence, all spaces of constant sectional curvature are Einstein and even more so every two-dimensional manifold is also Einstein.

2.3 Lie groups and Lie algebras

When a manifold is also endowed with a smooth group operation, it enables rich structure. We call such manifolds *Lie groups* and they can be brought into a relation to their tangent spaces at the identity which we call *Lie algebra*. Many of the standard matrix groups are examples of these special manifolds which means that we can cast a differential geometric light on them. In this section we review some of the core concepts and results in the study of Lie groups and their Lie algebras, which play a major role in many aspects of modern theoretical physics.

Definition 2.48. A smooth manifold G is called **Lie group**, if it is a group and the group structure is compatible with the manifold structure, that is, multiplication and inversion are smooth.

Example 2.49. The most prominent examples of Lie groups are the familiar matrix groups.

(i) $GL(\mathbb{K}, n) = \{X \mid \det X \neq 0\} \subset \mathbb{K}^{n\times n}$

(ii) $SL(\mathbb{K}, n) = \{X \mid \det X = 1\} \subset GL(\mathbb{K}, n)$

(iii) $O(p, q) = \{X \mid X^T \eta^{(p,q)} X = \mathbb{1}\} \subset GL(\mathbb{R}, n)$

(iv) $SO(p, q) = \{X \mid X^T \eta^{(p,q)} X = \mathbb{1}\,, \det X = 1\} \subset SL(\mathbb{R}, n)$

(v) $U(n) = \{X \mid X^\dagger X = \mathbb{1}\} \subset GL(\mathbb{C}, n)$

(vi) $SU(n) = \{X \mid X^\dagger X = \mathbb{1}\,, \det X = 1\} \subset SL(\mathbb{C}, n)$

Definition 2.50. Let G be a Lie group and $H \subset G$. We then call H **Lie subgroup** of G, if it is both a subgroup and the inclusion is an injective immersion. Furthermore we call a Lie subgroup **closed** if it is closed in G as a topological set.

Theorem 2.51 (Cartan)**.** *Let G be a Lie group and $H \subset G$ a Lie subgroup. Then, H is closed if and only if the inclusion is a smooth embedding.*

Definition 2.52. Let G be a Lie group and M a manifold. We then call a smooth map

$$L : G \times M \to M \tag{2.143}$$
$$(g, p) \mapsto L_g(p) \equiv gp \tag{2.144}$$

a **left action** if (i) $L_g \circ L_h = L_{gh}$ and (ii) $L_e = \mathrm{id}_M$. Likewise we call a smooth map

$$R : G \times M \to M \tag{2.145}$$
$$(g, p) \mapsto R_g(p) \equiv pg \tag{2.146}$$

a **right action** if (i) $R_g \circ R_h = R_{hg}$ and (ii) $R_e = \mathrm{id}_M$. Furthermore, for any action $\psi : G \times M \to M$ and $p \in M$ we call the set

$$\mathcal{O}_p := \{\psi(g, p) \mid g \in G\} \equiv \psi(G, p) \subset M \tag{2.147}$$

the **orbit of** p and the set

$$\mathrm{Stab}_p := \{g \in G \mid \psi(p, g) = p\} \subset G \tag{2.148}$$

the **stabilizer subgroup** of p, which is a closed Lie subgroup of G.

Definition 2.53. Let $\psi : G \times M \to M$ an action. We then call ψ

(i) **effective** if $\psi(g, \cdot) = id_M \ \Leftrightarrow \ g = e$

(ii) **free** if $\text{Stab}_p = \{e\} \ \forall p \in M$

(iii) **transitive** if $\forall p, q \in M \exists g \in G : p = \psi(q, g)$

(iv) **proper** if $(g, p) \mapsto (\psi(g, p), p)$ is proper

In particular every Lie group naturally acts on itself via left and right translation. From left and right translation we can define an additional left action of a Lie group on itself via conjugation.

Definition 2.54. Let G be a Lie group. Then, the **conjugation** is given by

$$\alpha : G \times G \to G \tag{2.149}$$

$$(g, h) \mapsto \alpha_g(h) := ghg^{-1} = L_g \circ R_{g^{-1}}(h) \tag{2.150}$$

which is a left action of G on itself.

For a fixed $g \in G$ all three actions L_g, R_g and α_g become diffeomorphisms of G into itself.

Definition 2.55. Let G act on some manifold M. We call an object f defined on M **invariant under** G **action** if $f(\psi(g, p)) = f(p) \ \forall g \in G$. Furthermore if f lies in a class of objects, which themselves also carry a G action $\bar{\psi}$, we call f **equivariant under** G **action** if the action pulls out, that is, $f(\psi(g, p)) = \bar{\psi}(g, f(p))$.

Definition 2.56. Let G be a Lie group. We then call a vector field $X \in \mathfrak{X}(G)$ **left invariant** if it is invariant under the push forward of the left translation.

$$L_{g*}X = X \ \forall g \in G \tag{2.151}$$

The set of all left invariant vector fields forms a Lie subalgebra of $\mathfrak{X}(M)$ and we denote it by $\text{Lie}(G)$ and we call it *the* **Lie algebra of** G.

The Lie algebra of a Lie group G is isomorphic to the tangent space at the identity T_eG. We see this by simply defining

$$\tilde{X}|_g := L_{g*}|_e X \tag{2.152}$$

for $X \in T_eG$, which is by construction left invariant. Due to this fact, we are led to the following definition.

Definition 2.57. Let G be a Lie group. We then denote the tangent space at the identity with

$$\mathfrak{g} := T_eG \cong \text{Lie}(G) \tag{2.153}$$

and also call it *the* **Lie algebra of** G. The tangent space at the identity inherits a Lie bracket by restricting the commutators of left invariant vector fields. Given a basis $\{I_a\}$ on $\mathfrak{g}$ we define the **structure constants of** $\mathfrak{g}$ $f^c_{\ ab}$ as

$$[I_a, I_b] = f^c_{\ ab} I_c. \tag{2.154}$$

Naturally, $\dim(\mathfrak{g}) = \dim(G)$.

Theorem 2.58 (Cartan)**.** *Let G be a Lie group and $H \subset G$ a closed Lie subgroup. Then, there exists a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ which is the Lie algebra of H.*

Definition 2.59. Let G be a Lie group. We then call a one-form $\omega \in \Omega^1(G)$ **left invariant** if it is invariant under the pull back of the left translation.

$$L_g^* \omega = \omega \ \forall g \in G \tag{2.155}$$

Naturally the space of left invariant one-forms is the dual of the Lie algebra $\text{Lie}(G)^* \cong \mathfrak{g}^*$.

In the case of left invariant dual frames $\{E_a\}, \{e^b\}$, the *Maurer-Cartan equations* hold

$$\mathrm{d}e^c = -\frac{1}{2} f^c{}_{ab} e^a \wedge e^b. \tag{2.156}$$

Since every left invariant vector field is generated by a corresponding element of $T_eG \cong \mathfrak{g}$, one can define a special one-form, which maps the vector field onto its generator.

Definition 2.60. Let G be a Lie group and $\{I_a\} \subset \mathfrak{g} \cong \{E_a\} \subset \mathrm{Lie}(G)$, $\{e^a\} \subset \mathrm{Lie}(G)^*$ left invariant dual frames. We then define the **Maurer-Cartan form** as

$$\omega := I_a \, e^a. \tag{2.157}$$

More abstractly we have that

$$\omega = g^{-1}\mathrm{d}g \;\text{ or }\; \omega|_g = DL_{g^{-1}}|_g. \tag{2.158}$$

It is thus a Lie algebra valued one-form on the group. From the Maurer-Cartan equations it follows immediately that

$$\mathrm{d}\omega = -\frac{1}{2}[\omega, \omega] \tag{2.159}$$

where the multiplication of forms is understood as the wedge product in the commutator.

Theorem 2.61. *Let $X \in \mathfrak{g}$ be a left invariant vector field and Θ_t be its flow. Then*

$$\Theta_t(g) = g\Theta_t(e) \tag{2.160}$$

which implies that the flow of X is global, i.e. defined on all of $\mathbb{R}$.

Definition 2.62. Let G be a Lie group. We then define the **exponential map** as

$$\exp : \mathfrak{g} \to G \tag{2.161}$$

$$X \mapsto \exp(X) \equiv e^X := \Theta_1(e) \tag{2.162}$$

where Θ_t is the flow of X.

The exponential map is one of the key objects in the study of Lie groups. One can show that it is surjective onto the connected component of the identity and also locally diffeomorphic, in fact, restricting a subset of the Lie algebra appropriately, it becomes diffeomorphic to the connected component of the identity. That is, the exponential map enables one to do group theory by doing linear algebra. The actual representation of the exponential map is not always easy to find but in the case of matrix groups (which we will always consider) it is given by the familiar matrix exponential. Furthermore for fixed $X \in \mathfrak{g}$ the set $\exp(\mathbb{R}X) \subset G$ is a one parameter Lie subgroup of G.

Another way of defining the Lie algebra of a Lie group is through the set of elements X such that $\exp(X) \in G$.[4] From this, one can easily determine the properties of the Lie algebras of the standard Lie groups, one finds the following list.

(i) $\mathfrak{gl}(\mathbb{K}, n) = \mathrm{End}(\mathbb{K}, n) = \mathbb{K}^{n \times n}$

(ii) $\mathfrak{sl}(\mathbb{K}, n) = \{X \mid \mathrm{tr}X = 0\} \subset \mathfrak{gl}(\mathbb{K}, n)$

(iii) $\mathfrak{o}(p, q) = \{X \mid X^T \eta^{(p,q)} + \eta^{(p,q)} X = 0\} \subset \mathfrak{gl}(\mathbb{R}, n)$

(iv) $\mathfrak{so}(p, q) = \mathfrak{o}(p, q)$

(v) $\mathfrak{u}(n) = \{X \mid X^\dagger = -X\} \subset \mathfrak{gl}(\mathbb{C}, n)$

(vi) $\mathfrak{su}(n) = \{X \mid X^\dagger = -X \,,\, \mathrm{tr}X = 0\} \subset \mathfrak{sl}(\mathbb{C}, n)$

[4] This does in general not yield the full group, though the portion one gets suffices most of the time.

Definition 2.63. Let G be a Lie group and α the conjugation. We then define the **adjoint representation of G on $\mathfrak{g}$** as

$$\begin{aligned} \mathrm{Ad} : G &\to GL(\mathfrak{g}) && (2.163)\\ g &\mapsto \mathrm{Ad}_g := D\alpha_g|_e. && (2.164) \end{aligned}$$

It is easy to check that Ad is actually a representation of G. In the case of matrix groups the adjoint representation of G is given by

$$\mathrm{Ad}_g(X) = gXg^{-1} \tag{2.165}$$

where the products are matrix multiplication.

Definition 2.64. Let G be a Lie group. We then define the **adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}$** as

$$\begin{aligned} \mathrm{ad} : \mathfrak{g} &\to \mathfrak{gl}(\mathfrak{g}) && (2.166)\\ X &\mapsto \mathrm{ad}(X) := D\mathrm{Ad}|_e(X) = [X, \cdot]. && (2.167) \end{aligned}$$

It indeed is a representation of $\mathfrak{g}$. Given a basis $\{I_a\}$, the adjoint representation is given by the structure constants $\mathrm{ad}(I_a)^b{}_c = f^c{}_{ab}$.

Theorem 2.65. *Let G be a Lie group. Then*

$$\mathrm{Ad}_{e^X} = e^{\mathrm{ad(X)}} \; \forall X \in \mathfrak{g}. \tag{2.168}$$

Since Lie groups are also manifolds, we can also talk about metrics. Of particular interest are metrics which are adapted to the group structure. It is straight forward to define a *left invariant metric* on a Lie group by simply defining an inner product on T_eG and the pushing it along the whole group via left translation. A question arising then is, when is such a metric also right invariant? That is, can there be (pseudo-)Riemannian metrics on G which are totally G invariant?

Theorem 2.66. *Let G be a Lie group and $\langle , \rangle$ a left invariant (pseudo-)Riemannian metric and ∇ the corresponding Levi-Civita connection. Then the following statements are equivalent.*

(i) $\langle , \rangle$ *is also right invariant*

(ii) $\langle , \rangle|_{\mathfrak{g}}$ *is* Ad_g *invariant*

(iii) The group inversion is an isometry

(iv) $\mathrm{ad}^\dagger = -\mathrm{ad}$

(v) $\nabla_X Y = \frac{1}{2}[X, Y]$

(vi) Geodesics starting at e are one parameter subgroups given by $\exp(tX)$

It turns out that we can always define a left invariant metric with property (ii),

Definition 2.67. Let G be a Lie group. We then call the **Killing form** on $\mathfrak{g}$ as

$$K(X, Y) := \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)) \equiv \mathrm{tr}_{\mathrm{adj}}(XY) \tag{2.169}$$

which is an Ad_g invariant, symmetric inner product on $\mathfrak{g}$. Furthermore we define via left translation the **Cartan-Killing metric** on G as

$$g|_h := L_{h*}|_e K. \tag{2.170}$$

Given a basis on the Lie algebra $\{I_a\}$ with corresponding dual left invariant one forms $\{e^a\}$ it reads

$$g = g_{ab}\, e^a \otimes e^b \;, \;\; g_{ab} = K(I_a, I_b). \tag{2.171}$$

We can express the Killing form in a given basis via the structure constants as

$$K_{ab} = f^c{}_{ad} f^d{}_{bc}. \tag{2.172}$$

Notice that the Killing form is a priori not non-degenerate or of definite signature. It turns out that the Killing form is non-degenerate if and only if the Lie algebra is *semisimple* which is known as the *Cartan criterion*, though we will not delve into the meaning of this.

Theorem 2.68. *A connected (real) Lie group is compact and semisimple if and only if its Killing form is negative definite.*

Hence, every compact, semisimple (real) Lie group can be made into a very symmetric, Riemannian manifold by using the negative of the Cartan-Killing metric. In this case, the Christoffel symbols are proportional to the structure constants, making the geometry of Lie groups very special and simple.

2.4 Homogeneous and symmetric spaces

Homogeneous spaces (and symmetric ones as a special case of the former) are spaces arising when (among other things) we mod out sub groups out of a Lie group. Through this they automatically become very symmetric, as Lie groups act as continuous transformations. Hence, in this section we look at homogeneous spaces and how objects on them relate to objects defined on Lie groups over them. As we will see, many of the ordinary highly symmetric manifolds are examples of these. This is the case especially for the classic constant curvature Riemannian and Lorentzian manifolds.

Definition 2.69. Let G be a Lie group and M a manifold on which G acts transitively. We then call the pair (M, G) **homogeneous space**.

Theorem 2.70. *Let (M, G) be a homogeneous space, where the action is also smooth. Then all stabilizers are isomorphic $\mathrm{Stab}_p \cong \mathrm{Stab}_q \cong: H\ \forall p, q \in M$. Furthermore if the stabilizer subgroup $H \subset G$ is a closed Lie subgroup of G, then the manifold is isomorphic to the right coset $M \cong G/H$. Conversely, when $H \subset G$ is a closed Lie subgroup it follows that $M := G/H$ has a unique smooth structure, making (M, G) a homogeneous space (via left action). Furthermore in both cases the natural projection $\pi : G \to G/H$ becomes a smooth submersion and it follows straight forwardly that $\dim(M) = \dim(G) - \dim(H)$.*

From now on, when talking about homogeneous spaces, we will always mean those which can indeed be written as cosets as described in the theorem. Henceforth we will use the terms homogeneous space and coset space interchangeably.

A question arising naturally is if it is possible to equip a coset space with a (pseudo-)Riemannian metric which is invariant under the action of the group. That is, the group action is an isometry. This is indeed a big topic in the study of Lie groups and in general dealt with by considering the so called *isotropy representation* of a coset. For our intents and purposes it is enough to consider a special class of cosets where the questions becomes more approachable.

Definition 2.71. Let G/H be a coset space. We then call the coset **reductive** if the Lie algebra of G decomposes

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \tag{2.173}$$

such that $\mathfrak{m}$ is Ad_H invariant, that is,

$$\mathrm{Ad}_H(\mathfrak{m}) \subset \mathfrak{m} \ \Leftrightarrow\ \mathrm{ad}(\mathfrak{h})(\mathfrak{m}) \equiv [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}. \tag{2.174}$$

For a reductive homogeneous space we introduce new notation. We index a basis of the Lie algebra $\mathfrak{g}$ with capital letters, i.e.

$$\mathfrak{g} = \mathrm{span}\{I_A\}_{A=1,\dots,\dim(G)} \tag{2.175}$$

hence, we have

$$[I_A, I_B] = f^C_{\ AB}\, I_C. \tag{2.176}$$

The splitting of the algebra then introduces a splitting of generators. We index generators from the sub algebra $\mathfrak{h}$ with greek indices and the generators from the complement $\mathfrak{m}$ with small latin ones.

$$\mathfrak{h} = \mathrm{span}\{I_\alpha\} \tag{2.177}$$
$$\mathfrak{m} = \mathrm{span}\{I_b\} \tag{2.178}$$

Reductivity of the coset then translates into the structure constants

$$[I_\alpha, I_\beta] = f^\gamma_{\ \alpha\beta}\, I_\gamma \tag{2.179}$$
$$[I_\alpha, I_a] = f^b_{\ \alpha a}\, I_b \tag{2.180}$$
$$[I_a, I_b] = f^c_{\ ab}\, I_c + f^\alpha_{\ ab}\, I_\alpha \tag{2.181}$$

that is, $f^\beta_{\ \alpha b} \equiv 0$. Furthermore it is easy to see that in this case $\dim(G/H) = \dim(\mathfrak{m})$ and even more so one can show that $T_{[e]}G/H \cong \mathfrak{m}$, hence we call the elements in $\mathfrak{m}$ the ‘coset generators’.

Theorem 2.72. *Let G/H be a reductive coset space. Then there is a one to one correspondence between G-invariant (pseudo-)Riemannain metrics on G/H and Ad_H-invariant non-degenerate products $\langle,\rangle_{\mathfrak{m}}$ on $\mathfrak{m}$. In particular, if the representation $\mathrm{Ad} : H \to GL(\mathfrak{m})$ is irreducible, this metric will be unique up to constant multiple. Furthermore by extending the inner product $\langle,\rangle_{\mathfrak{m}}$ to any non-degenerate inner product $\langle,\rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ such that the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ becomes orthogonal, we get that the natural projection $\pi : G \to G/H$ is a Riemannian submersion with respect to the left invariant metric induced by $\langle,\rangle_{\mathfrak{g}}$.*

Theorem 2.73. *Let G/H be a coset space where G is connected and $\langle,\rangle$ an Ad_G-invariant non-degenerate inner product on $\mathfrak{g}$. Furthermore denote with $\mathfrak{m}$ the orthogonal complement of $\mathfrak{h}$ with respect to the invariant product. Then, G/H is reductive with respect to the orthogonal decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.*

Normally one talks about G-invariant *Riemannian* metrics on coset spaces. In that case the theorems above actually become quite restrictive since the existence of these metrics is dependent on the existence of Ad invariant *positive definite* products. These, of course, do not always exist. But since we relax the condition, as we are going to deal with pseudo-Riemannian manifolds, the theorems actually immediately imply the existence of G-invariant metrics on cosets simply by using the Killing form as long as the Killing form is non-degenerate, which is the case if and only if G is semisimple.

Definition 2.74. Let $M := G/H$ be a coset space. Then every element $X \in \mathfrak{g}$ induces a vector field $\tilde{X} \in \mathfrak{X}(M)$ via

$$\tilde{X}|_p := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \exp(tX)p \tag{2.182}$$

if M is equipped with a G-invariant metric g, we call these fields $\tilde{X}$ **Killing vector field**. Of course this is because they are Killing in the sense as we have defined it before, that is

$$\mathcal{L}_{\tilde{X}} g \equiv 0. \tag{2.183}$$

Theorem 2.75. *Let $M := G/H$ be a coset space with G-invariant metric. Then $\dim(G) \leq \frac{1}{2}n(n+1)$, where n is the dimension of the coset. Furthermore if the upper bound is attained, the coset is maximally symmetric.*

Definition 2.76. Let G/H be a reductive coset space with decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. We then call it **symmetric space** if

$$[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}. \tag{2.184}$$

That is, the Lie algebra decomposes such that

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}\ ,\ [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}\ ,\ [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h} \tag{2.185}$$

or equivalently if

$$[I_\alpha, I_\beta] = f^{\gamma}_{\ \alpha\beta}\, I_\gamma \tag{2.186}$$

$$[I_\alpha, I_a] = f^{b}_{\ \alpha a}\, I_b \tag{2.187}$$

$$[I_a, I_b] = f^{\alpha}_{\ ab}\, I_\alpha \tag{2.188}$$

that is, $f^{\beta}_{\ \alpha b} \equiv 0$ and $f^{c}_{\ ab} \equiv 0$. We call such a decomposition **Cartan decomposition**.

Finally, lets have a look at the geometry of symmetric spaces and how the previous discussions play out in practise. We will only consider Lie groups G which are connected and semisimple. If G is not connected by itself, we will just restrict ourselves to the connected component of the identity G_e which we will still simply denote with G. Thus, for a closed Lie subgroup H we have our symmetric spaces given by G/H whose dimension we set to n. Since G is semisimple, the Killing form will be non-degenerate and hence provides us the orthogonal Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}. \tag{2.189}$$

As before we denote with

$$\{I_A\} = \{I_\alpha\} \cup \{I_a\} \subset \mathfrak{g} \tag{2.190}$$

a Killing-orthogonal basis of the Lie algebra. Dual to the left invariant vectorfields $\{\hat{E}_A\} \subset \mathrm{Lie}(G)$ generated by the elements $\{I_A\}$ on G, we have a set of orthogonal, left invariant one forms $\{\hat{e}^A\} \subset \mathrm{Lie}(G)^*$. Hence, the Cartan-Killing metric in this basis becomes

$$g = K(I_A, I_B)\, \hat{e}^A \otimes \hat{e}^B = K_{AB}\, \hat{e}^A \otimes \hat{e}^B \tag{2.191}$$

that is

$$g_{AB} = K_{AB} = f^C_{\ AD} f^D_{\ BC}. \tag{2.192}$$

Since we are dealing with a symmetric space, we have the following commutation relations

$$[I_\alpha, I_\beta] = f^\gamma_{\ \alpha\beta}\, I_\gamma\ ,\ \ [I_\alpha, I_a] = f^b_{\ \alpha a}\, I_b\ ,\ \ [I_a, I_b] = f^\alpha_{\ ab}\, I_\alpha. \tag{2.193}$$

Which implies for the for the metric components

$$g_{\alpha\beta} = f^\gamma_{\ \alpha\delta} f^\delta_{\ \beta\gamma} + f^b_{\ \alpha a} f^a_{\ \beta b}\ ,\ \ g_{ab} = 2 f^\alpha_{\ ad} f^d_{\ b\alpha}\ ,\ \ g_{\alpha b} = 0. \tag{2.194}$$

With this we can now equip the coset with a G invariant metric. To this end we consider a smooth map $\sigma : G/H \to G$ with the property $\pi \circ \sigma = \mathrm{id}$. That is, for each point $p \in G/H$ it picks out one representative in G. Using σ we can now pull back the left invariant one forms on G, we distinguish the pulled back forms by leaving out the hat.

$$e^A := \sigma^* \hat{e}^A \in \Omega^1(G/H) \tag{2.195}$$

Naturally, since $\dim(G/H) = \dim(\mathfrak{m})$ the forms will be ($\mathcal{C}^\infty$-)linearly dependent.

$$e^\alpha = \chi^\alpha_{\ b} e^b\ ,\ \ \chi^\alpha_{\ b} \in \Omega^0(G/H) \tag{2.196}$$

To obtain the G-invariant metric on the coset, we simply pull back the restriction of the Cartan-Killing metric to the coset generators, that is

$$g_{G/H} = \sigma^* g|_{\mathfrak{m}} = g_{ab}\, e^a \otimes e^b \tag{2.197}$$

With this, we have turned the symmetric spaces G/H into a (pseudo-)Riemannian manifold on which G acts isometrically. Now consider the fact that the forms e^A come from a left invariant dual frame on G. That means that they obey the Maurer-Cartan equations

$$\mathrm{d}e^A = -\frac{1}{2} f^A_{\ BC} e^B \wedge e^C. \tag{2.198}$$

which will split up due to the fact that G/H is symmetric

$$\mathrm{d}e^a + f^a_{\ \alpha b} \chi^\alpha_{\ c}\, e^c \wedge e^b = 0\ ,\ \ \mathrm{d}e^\alpha + \frac{1}{2} f^\alpha_{\ AB} e^A \wedge e^B = 0. \tag{2.199}$$

In addition we have on the coset the Cartan structure equations for the Levi-Civita connection

$$\mathrm{d}e^a + \Gamma^a_{\ b} \wedge e^b = 0. \tag{2.200}$$

Hence, by comparing the two, we can immediately write down the Christoffel symbols

$$\Gamma^a_{\ b} = \Gamma_{c\ b}^{\ a}\, e^c = f^a_{\ \alpha b} \chi^\alpha_{\ c} e^c. \tag{2.201}$$

Likewise the curvature tensor follows from the second structure equation

$$R = \mathrm{d}\Gamma + \Gamma \wedge \Gamma \tag{2.202}$$

we obtain the components of the curvature tensor as

$$R^a_{\ bcd} = -\frac{1}{2} f^a_{\ \gamma b} f^\gamma_{\ cd} + (-f^a_{\ \ell\alpha} f^\ell_{\ \beta b} - \frac{1}{2} f^a_{\ \gamma b} f^\gamma_{\ \alpha\beta}) \chi^\alpha_{\ c} \chi^\beta_{\ d}. \tag{2.203}$$

Finally from contractions with the metric we obtain the Ricci- tensor and scalar

$$R_{bd} = -\frac{1}{4} K_{bd} + (-f^a_{\ \ell\alpha} f^\ell_{\ \beta b} - \frac{1}{2} f^a_{\ \gamma b} f^\gamma_{\ \alpha\beta}) \chi^\alpha_{\ a} \chi^\beta_{\ d} \tag{2.204}$$

$$\mathcal{R} = -\frac{1}{4} K^a_{\ a} + (-f^a_{\ \ell\alpha} f^\ell_{\ \beta b} - \frac{1}{2} f^a_{\ \gamma b} f^\gamma_{\ \alpha\beta}) \chi^\alpha_{\ a} \chi^{\beta b}. \tag{2.205}$$

As they will be of our interest later on, we conclude the section with a list of properties for the four classic constant sectional curvature Riemannian and Lorentzian manifolds. For the decompositions G/H we chose the symmetric cosets but there are also alternatives. The metrics of the spaces are abstractly defined as the restriction of the respective ambient metrics in $\tilde{\eta}^{(p,q)}$ to the tangent bundle, which are the same as the ones obtained from the Cartan-Killing metric.

M	G/H	Signature	k
Sphere $S^n = \{\vec{x} \in \mathbb{R}^{n+1} \mid \vec{x}^T \eta^{(n+1,0)} \vec{x} = 1\}$	$SO(n+1)/SO(n)$	Riemannian	$+1$
Hyperbolic space $H^n = \{\vec{x} \in \mathbb{R}^{n+1} \mid \vec{x}^T \eta^{(1,n)} \vec{x} = 1\}$	$SO(1,n)/SO(n)$	Riemannian	-1
de Sitter space $dS_n = \{\vec{x} \in \mathbb{R}^{n+1} \mid \vec{x}^T \eta^{(1,n)} \vec{x} = -1\}$	$SO(1,n)/SO(1,n-1)$	Lorentzian	$+1$
Anti-de Sitter space $AdS_n = \{\vec{x} \in \mathbb{R}^{n+1} \mid \vec{x}^T \eta^{(2,n-1)} \vec{x} = 1\}$	$SO(2,n-1)/SO(1,n-1)$	Lorentzian	-1

2.5 General Relativity

In this section we briefly cover some concepts of the theory of general relativity (GR). We will keep the discussion at the bare necessities since this work is not focused on GR. In particular we are only interested in the general idea of spacetime as a Lorentzian manifold and a particular prominent family of solutions found in the field of cosmology.

Definition 2.77. An smooth, orientable, Lorentzian manifold (M, g) is called **spacetime**. In this context we enumerate coordinate- and tensor components with greek letters $\mu = 0, ..., \dim(M) - 1$. Working in mostly minus signature, that is, in an orthonormal frame the metric tensor becomes $g_{\mu\nu} = \text{diag}(+1, -1, ..., -1)$, we call a tangent vector $v \in T_pM$ (i) **timelike** if $g(v, v) > 0$, (ii) **spacelike** if $g(v, v) < 0$ and **lightlike** if $g(v, v) = 0$. A vector field has the same attributes if it obeys these relations at every point. The same goes for coordinates depending on their induced tangent vector fields; we usually arrange it in such a fashion that the 0-component is timelike. We also give a submanifold these attributes if all of its tangent spaces obey these relations. Notice that the dimension is arbitrary (though it of course needs to be bigger than one); normally when doing general relativity the dimension is fixed to four but generalizations are possible.

Definition 2.78. Let (M, g) be a spacetime and ∇ its Levi-Civita connection. We then define the **Einstein tensor** $G \in \Gamma(T^{(0,2)}M)$ as

$$G := Ric - \frac{1}{2}\mathcal{R}g \tag{2.206}$$

that is,

$$G_{\mu\nu} = Ric_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} \tag{2.207}$$

which, in four dimensions, is nothing but the *trace reversal* of the Ricci tensor $G = Ric - \frac{1}{2}\text{tr}_g(Ric)g$. The Einstein tensor obeys

$$G_{\mu\nu} = G_{\nu\mu} \tag{2.208}$$
$$\nabla_\mu G^\mu_{\ \nu} \equiv 0. \tag{2.209}$$

Indeed, for four dimensional space times it can be shown that *any* rank (0,2) tensor which (i) depends at most on second derivatives of g and (ii) is covariantly divergence free, must be of the form $aG + bg$ for $a, b \in \mathbb{R}$, making it automatically symmetric. This fact is known as *Lovelocks Theorem.*

Definition 2.79. A tensor field $T \in \Gamma(T^{(0,2)}M)$ on a spacetime is called **Energy-momentum tensor** if it is both symmetric and covariantly divergence free, that is

$$T_{\mu\nu} = T_{\nu\mu} \tag{2.210}$$
$$\nabla_\mu T^\mu_{\ \nu} \equiv 0 \tag{2.211}$$

i.e. it has the same properties as the Einstein tensor. In an orthonormal frame $\{e_\mu\}$ the components of T can be interpreted as the following physical quantities as they are seen by an observer whose rest frame is given by the $\{e_\mu\}$

$$T_{\mu\nu} = \begin{pmatrix} \mathcal{E} & -c\vec{\mathcal{P}}^T \\ -\frac{1}{c}\vec{\mathcal{S}} & \mathcal{M}_{ab} \end{pmatrix} \tag{2.212}$$

where $\mathcal{E}$ is energy density, $\vec{\mathcal{P}}$ is momentum density, $\vec{\mathcal{S}}$ energy current density and $\mathcal{M}_{ab}$ is momentum current density. Notice that this interpretation together with the necessity of covariant divergencelessnes automatically yields the corresponding local conservation laws of these quantities.

Definition 2.80. Let (M, g) be a spacetime and T an energy-momenutm tensor. We then define the **Einstein equations** as

$$G + \Lambda g = \kappa T \tag{2.213}$$

where $\kappa := \frac{8\pi G_N}{c^4}$ is the *Einstein gravitational constant*[5] (G_N is Newton's constant) and $\Lambda \in \mathbb{R}$ is called **cosmological constant**. There is the possibility to set $\Lambda = 0$, we then speak of the Einstein equations without cosmological constant.

The Einstein equations capture the idea that matter determines spacetime curvature and (*simultaneously*) that curvature determines the motion of matter. There are different ways to pose the problem. If we provide a fixed energy momentum tensor, the equations become a system of non-linear, second order PDE's for the components of the metric tensor. If we instead also leave the energy momentum tensor mostly undetermined, the system becomes even more complicated. Hence, in general there is some further structure or symmetry imposed on both T and g thus reducing the degrees of freedom substantially. We will shortly see an example, where the system is reduced to the minimal possible degrees of freedom. Furthermore note that in the special case of four spacetime dimensions, the Einstein tensor (or generally the left hand side of the equations if we fix Λ) has 10 independent components, that is exactly half of the number of independent components of the Riemann tensor in this case[6]. This means in four dimensional GR, only half of the curvature components are actually determined by the matter distribution, thus allowing for vacuum dynamics ($T = 0$) of the spacetime, like e.g. gravitational waves; In this case we speak of the *vacuum* Einstein equations.

Remark 2.81. Consider the vacuum Einstein equations

$$G + \Lambda g = 0. \tag{2.214}$$

If the spacetime is an Einstein manifold, i.e. $Ric = k\, g$, of dimension $n > 2$ we have

$$G = Ric - \frac{1}{2}\mathcal{R}g + \Lambda g = \left(\frac{k(2-n)}{2} + \Lambda\right) g = 0 \tag{2.215}$$

Thus every such Einstein manifold is automatically a solution to the vacuum Einstein equations for $\Lambda = \frac{k(n-2)}{2}$.

Remark 2.82. Given a solution of the Einstein equations (M, g), a particle moves along the geodesics of the Levi-Civita connection. That is, the action of a point particle is the length functional. In this context we call the Lorentzian length of a path in spacetime the *eigentime*.

The Einstein equations can also be obtained by the variation of an action functional. In this way it is also possible to obtain a 'canonical' energy-momentum tensor of a given matter field, let us see how this plays out.

Definition 2.83. Let (M, g) be a spacetime. We then define the **Einstein-Hilbert action** as

$$S[g] := \frac{1}{2\kappa}\int_M \mathrm{dVol}(\mathcal{R} - 2\Lambda) = \frac{1}{2\kappa}\int_M \mathrm{d}^n x\sqrt{-g}(\mathcal{R} - 2\Lambda). \tag{2.216}$$

The variation of the Einstein-Hilbert action with respect to the (inverse) metric tensor yields precisely the vacuum Einstein equations, that is

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0 \;\Leftrightarrow\; G + \Lambda g = 0. \tag{2.217}$$

Furthermore given an action for some matter field, that is, a Lagrangian $\mathcal{L}_{\text{matter}}(\phi, \partial_\mu \phi)$, we define the action of the coupled system as

$$S[g, \phi] := \int_M \left(\frac{1}{2\kappa}(\mathcal{R} - 2\Lambda) + \mathcal{L}_{\text{matter}}\right)\sqrt{-g}\mathrm{d}^n x. \tag{2.218}$$

In this case variation with respect to the metric yields

$$G + \Lambda g = \kappa T \tag{2.219}$$

[5]Unless otherwise stated we will always set $c = 1$ from now on.

[6]The rest resides in the *Weyl tensor*.

where T is now defined as the variation of the matter part in S with respect to the (inverse) metric, we have in general

$$T_{\mu\nu} = 2\frac{\delta \mathcal{L}_{\text{matter}}}{\delta g^{\mu\nu}} - g_{\mu\nu}\mathcal{L}_{\text{matter}} \tag{2.220}$$

Note however that in this case we can (and must) also vary the action with respect to the matter field, which in addition to the Einstein equations will give us yet another set of equations, namely the (covariant) equations of motion of our matter field(s) ϕ. Amazingly these become the ordinary Euler-Lagrange equations of ϕ where all differentiations are replaced by covariant derivatives, we have

$$\nabla_\mu \frac{\partial \mathcal{L}_{\text{matter}}}{\partial(\nabla_\mu \phi)} - \frac{\mathcal{L}_{\text{matter}}}{\partial \phi} = 0 \tag{2.221}$$

Hence, in this case the system becomes yet even more complicated to solve.

An important example of the latter is a self interacting scalar field

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi). \tag{2.222}$$

For which the equations of motion become

$$\nabla_\mu \partial^\mu \phi + V'(\phi) = 0 \tag{2.223}$$

and the corresponding energy-energy momentum tensor reads

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2 + g_{\mu\nu}V(\phi). \tag{2.224}$$

After having introduced the basic notions needed to work with GR, we now come to a certain class of special solutions. As one might have already guessed, without further restrictions imposed on the metric and energy-momentum tensor, the equations become unapproachable. Hence, we now want to force the solution to have certain symmetries.

Definition 2.84. A four dimensional spacetime (M, g) is said to be **homogeneous and isotropic** if it decomposes into a warped cylinder $M \cong \mathbb{R} \times \tilde{M}$ with

$$g = \mathrm{d}t \otimes \mathrm{d}t - a^2(t)\tilde{g} \tag{2.225}$$

where

$$\tilde{g} = \frac{1}{(1 + \frac{k}{4}\vec{x}^2)} \mathrm{d}\vec{x}\dot{\otimes}\mathrm{d}\vec{x} \tag{2.226}$$

and $k \in \{+1, 0, -1\}$. That is, the spacetime is a *spacelike slicing* with Riemannian manifolds of constant (normalized) sectional curvature $k \in \{+1, 0, -1\} \leftrightarrow \tilde{M} \in \{S^3, \mathbb{R}^3, H^3\}$. We also call these metrics **Friedman-Lemaître-Robertson-Walker** (FLRW) metrics. The warping function $a(t)$ we call in this context the **scale factor**. Furthermore we also say that these kinds of spacelike slicings, are adapted to the **cosmological principle**. Finally note that all homogeneous and isotropic spactimes have an isometric $SO(3)$ action with spacelike orbits.

The FLRW spacetimes are models for the universe at cosmological scales (i.e. the whole universe). We see that the metric in this case is fixed up to one degree of freedom, the scale factor. Hence, in cosmology the universe can only change the size of its spatial part. The matter in these cosmological models is also of special type.

Definition 2.85. Let (M, g) be a FLRW spacetime. We call an energy-momentum tensor T of **perfect fluid type** if its components in an orthonormal frame read

$$T_{\mu\nu} = \begin{pmatrix} \rho & \\ & p\delta_{ab} \end{pmatrix} \tag{2.227}$$

that is

$$T = \rho g_{00}\, e^0 \otimes e^0 - p g_{ab}\, e^a \otimes e^b \tag{2.228}$$

for some functions $\rho = \rho(t)$, $p = p(t)$ called *energy density* and *pressure* respectively. Furthermore we call an equation

$$p = p(\rho) \tag{2.229}$$

an **equation of state**. Two important equations of state are given by $p \equiv 0$ which we call **dust** and $p = \frac{1}{3}\rho$ which we call **radiation**. Note that a perfect fluid radiation type energy-momentum tensor is traceless.

We remark that the definition of a perfect fluid is actually a bit different, though the structure is the same. But since we are not really interested in cosmology and only work with the mathematical structure, the above definition is enough for our later intentions. Now, in this setup the Einstein equations become remarkably simple. First of all we have that the Einstein tensor of a FLRW spacetime in an orthonormal frame is given by

$$G_{00} = \frac{3}{a^2}\left(\dot{a}^2 + k\right) \tag{2.230}$$

$$G_{ab} = -\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right)\delta_{ab} \tag{2.231}$$

where the dot denotes ∂_t. Hence together with a perfect fluid tensor we obtain as the Einstein equations

$$\ddot{a} = -\frac{\kappa}{6}a\left(\rho + 3p\right) + \frac{\Lambda}{3}a \tag{2.232}$$

$$\dot{a}^2 = \frac{\kappa}{3}\rho a^2 - k + \frac{\Lambda}{3}a^2 \tag{2.233}$$

we call these equations the **Friedmann equations**. Actually there is the possibility to derive yet another equation, namely

$$(a^3\rho)^{\cdot} + (a^3)^{\cdot}p = 0. \tag{2.234}$$

One can show that any two of these three equations always imply the other. The bare Friedmann equations are underdetermined by themselves but if we additionally impose an equation of state we get a well defined system of ODE's. Another important quantity often used in this context is the **Hubble parameter** defined as the logarithmic derivative of the scale factor

$$H(t) := \frac{\dot{a}}{a}. \tag{2.235}$$

Finally, let us go back to the scalar field and put it into an FLRW spacetime. We impose that the field is also homogeneous (and isotropic), that is, $\phi = \phi(t)$. In this case the energy-momentum tensor from before simplifies in an orthonormal frame to

$$T_{00} = \frac{1}{2}\dot{\phi}^2 + V(\phi) =: \rho \tag{2.236}$$

$$T_{ab} = \left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right)\delta_{ab} =: p\delta_{ab} \tag{2.237}$$

which is of perfect fluid type. Furthermore the equations of motion then become

$$\ddot{\phi} + V'(\phi) = -3H(t)\dot{\phi} \tag{2.238}$$

which are of the form of a one-dimensional Newtonain particle subject to the potential V and a so called **Hubble friction** $\sim H\dot{\phi}$.

3 Gauge field Theory

In this chapter we give an introductory summary to mathematical structure of gauge field theory and Yang–Mills theory. Since gauge theories are formulated in the differential geometric language of bundles, we will give a review of the latter in the first section. There we cover the main definitions and results of vector- , principal- and associated bundles. In the second section we introduce the notion of connections and curvature on principal bundles which play the roles of potentials and field strengths in physical theories. Finally, in the third section we review Yang–Mills theory and how it relates to physics. The theory of principal bundles and gauge fields lies at the heart of some of the best physical theories we have, including the standard model of particle physics and more modern formulations of general relativity. Hence, these formalisms are an indispensable tool for modern theoretical physics. The chapter is based on [18], [19], [13], [20], [21], [22].

3.1 Vector-, Principal- and Associated Bundles

The bundle formalism in differential geometry is a way to generalize ideas from ordinary differential geometry. A key idea in this is the 'geometrization' of maps on manifolds by combining the manifolds and the target space of the map into a new manifold. Maps then become subsets of this composite object. One can think of this as a generalization of taking the graph of a function[7]. Introducing Lie group actions into the game leads one naturally to a special class of bundles called *principal bundles*. As we will see, many of the objects we have encountered in ordinary differential geometry are special cases of the generalizations which we will develop in the following two sections.

We start of by defining the most basic type of bundle.

Definition 3.1. Let E, F and M be smooth manifolds and $\pi : E \to M$ a surjection. The triple (E, π, M) is called **locally trivial fibration with fiber** F if for every $m \in M$ there exists an open neighborhood $m \in U \subset M$ and a diffeomorphism $\phi : \pi^{-1}(U) \to U \times F$ such that $\mathrm{pr}_1(\phi(p)) = \pi(p)\ \forall p \in \pi^{-1}(U)$. In this case we call E the **total space**, M the **base space**, $\pi^{-1}(m) \cong F$ the **fiber** over $m \in M$ and ϕ a local **trivialization**.

In other words, a locally trivial fibration - or fiber bundle - is a collection of manifolds and maps such that the following diagram commutes.

$$\begin{array}{ccc} \pi^{-1}(U) \subset E & \xrightarrow{\quad\phi\quad} & U \times F \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & U \subset M & \end{array}$$

Given the definition of a fiber bundle, we infer that the tangent spaces at each point can be split $T_pE = T_p\mathrm{Vert} \oplus T_p\mathrm{Hor}$. Where we distinguish between the *vertical subspace* $T_pVert := \ker D\pi|_p$ and the rest $T_p\mathrm{Hor}$ - the *horizontal subspace*. We call vectors in each subspace vertical and horizontal respectively. So, the vertical directions point into the direction of the fiber at a point while the horizontal directions point along the fibers at a point. Furthermore, we call a fiber bundle *globally trivial* if there exists a global trivialization, that is $U = M$. A special case of the latter are the *trivial products* $E = M \times F$ with $\phi = \mathrm{id}_{M\times F}$ and $\pi = \mathrm{pr}_1$.

Definition 3.2. A locally trivial fibration of fiber type $F \cong \mathbb{K}^n$ is called $\mathbb{K}$**-Vector bundle of rank** n if

(i) every fiber $\pi^{-1}(m)$ is a n-dimensional $\mathbb{K}$ vector space

(ii) the second component of the trivialization restricted to a fiber $\phi_2|_{\pi^{-1}(m)} : \pi^{-1}(m) \to \mathbb{K}^n$ is $\mathbb{K}$ linear.

The fact that the trivialization is a diffeomorphism implies that ϕ_2 is a point wise vector space isomorphism.

Example 3.3. Let M be a smooth, n dimensional manifold. Then the Tangent bundle TM and Cotangent bundle T^*M are both $\mathbb{R}$-Vector bundles of rank n over M. Given a coordinate chart on

[7] For a map $x \mapsto f(x)$ one defines the graph as the collection of tupels $(x, f(x))$.

M, the induced trivializations are

$$\begin{aligned} \phi_{TM}\left(X|_m\right) &:= \left(m, \mathrm{d}x^1|_m\left(X\right), ..., \mathrm{d}x^n|_m\left(X\right)\right) \\ \phi_{T^*M}\left(\omega|_m\right) &:= \left(m, \omega|_m\left(\partial_{x^1}\right), ..., \omega|_m\left(\partial_{x^n}\right)\right) \end{aligned} \tag{3.1}$$

whose second (last n) components, of course, are point wise linear. Likewise all tensor bundles $T^{(r,s)}\left(M\right)$ and the k-form bundles are also vector bundles of respective ranks, the trivializations of which are constructed analogously to the above.

Definition 3.4. Let (E, π, M) be a locally trivial fibration and $U \subset M$ an open subset. We then call a map $\sigma : U \to E$ a **section** of the bundle if

$$\pi \circ \sigma = \mathrm{id}_U. \tag{3.2}$$

In case $U = M$ we call the section **global**, else **local**. Furthermore we denote the space of all sections $\Gamma\left(E\right)$.

Hence, all tensor fields, including vector fields and differential forms, are sections of their respective bundles.

Definition 3.5. Let G be a Lie group. A locally trivial fibration (P, π, M) with fiber type G is called **Principal G-bundle** over M if

(i) G acts freely from the right on P

(ii) $\pi(p) = \pi(q) \;\Leftrightarrow\; \exists g \in G \;:\; p = qg \;\forall p, g \in P$

(iii) there exist trivializations covering P which are G-equivariant, that is, $\phi_2(pg) = \phi_2(p)g$

In other words, having a principal bundle means that the following diagram commutes.

$$\begin{array}{ccc} \pi^{-1}(U) \subset P & \xrightarrow{\phi} & U \times G \\ {\scriptstyle R_g}\uparrow & & {\scriptstyle R_g}\uparrow \\ \pi^{-1}(U) \subset P & \xrightarrow{\phi} & U \times G \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & U \subset M & \end{array}$$

Where the R_g denotes the right action of the group on P and on itself respectively and that additionally the fibers are precisely the orbits under the right action. In this case we call G the **structure group** of the principal bundle.

Example 3.6.

- The *trivial* principal G-bundle over M (P, π, M) is defined by the product of G and M.

$$P := M \times G\ ,\ \pi := \mathrm{pr}_1\ ,\ \phi := \mathrm{id}_{M\times G}\ ,\ R_g(p) := (p_1, p_2 g) \tag{3.3}$$

- Let M be a n dimensional, smooth manifold. Consider the total space constructed by union of all bases at each tangent space of M, that is,

$$P := \{(m, v_1, ..., v_n) = (m, \vec{v}) \mid m \in M\ , \{v_i\} \text{ basis of } T_m M\} \tag{3.4}$$

together with the projection $\pi((m, \vec{v})) := m$. We can define a right $GL(\mathbb{R}, n)$ action via

$$R_T((m, \vec{v})) := (m, \vec{v}T) = (m, v_k T^k_{\ 1}, ..., v_k T^k_{\ n}) \tag{3.5}$$

thus turning (P, π, M) into a $GL(\mathbb{R}, n)$ principal bundle. This example is quite special and is also referred to as the *frame bundle* of M and denoted by $L(M)$. Furthermore, if M is orientable, we can construct a similar bundle by taking only oriented bases into account and acting with $SL(\mathbb{R}, n)$.

- Every homogeneous space G/H defines a principal H-bundle $P = G \cong G/H \times H$ with π being the natural projection.

Since on a principal bundle the group action is transitive in the fibers, one obtains the following result.

Theorem 3.7. *A principal bundle is globally trivial if and only if it admits a global section.*

Definition 3.8. Let (P, π, M) be a G principal bundle and N a smooth manifold on which G acts from the left. Furthermore let G act from the right on the product $P \times N$ via

$$R_g((p, n)) := (R_g(p), L_{g^{-1}}(n)) \tag{3.6}$$

and denote with $P \times_G N := (P \times N)/G$ the coset, whose elements we write as $[p, n]$. Finally define $\hat{\pi} : P \times_G N \to M$ via

$$\hat{\pi}([p, n]) := \pi(p). \tag{3.7}$$

We then call the triple $(P \times_G N, \hat{\pi}, M)$ an to (P, π, M) **associated bundle with fiber** N.

Theorem 3.9. *Let (P, π, M) be a principal G bundle, then an associated bundle $(P \times_G N, \hat{\pi}, M)$ is a locally trivial fibration.*

Example 3.10.

- Let (P, π, M) be a principal G bundle. Furthermore let V be a vector space over $\mathbb{K}$ and $\rho : G \to GL(V)$ be a representation. Hence, G acts on V from the left via $R_g = \rho(g)$ and thus with the definitions from above $P \times_G V$ is a associated bundle. In cases like this, where a representation ρ of the group is involved, we also write $P \times_\rho V$. Furthermore, together with

$$\begin{aligned} [p, v] + [p, w] &:= [p, v + w] \\ \lambda[p, v] &:= [p, \lambda v] \ , \ \lambda \in \mathbb{K} \end{aligned} \tag{3.8}$$

  $P \times_\rho V$ even becomes an associated vector bundle.

- It can be shown that every tensor and tensor density bundle over a manifold M is associated to the frame bundle $L(M)$ of M.

Theorem 3.11. *There is a one to one correspondence between the set of sections $\Gamma(P \times_G N)$ of an associated bundle with the set of maps $\lambda : P \to N$ with*

$$\lambda(R_g(p)) = L_{g^{-1}}(\lambda(p)). \tag{3.9}$$

Definition 3.12. Let (E, π, M) be a vector bundle. We then call

$$\Omega^k(M, E) := \Gamma\left(E \otimes \Lambda^k T^* M\right) \tag{3.10}$$

the space of **k-forms on M with values in E**. Effectively what is happening is that we enlarge the fibers of the k-form bundle by tensoring the fibers of E. Hence, everything is to be understood point wise, i.e. we can expand the sections as linear combinations of tensor products of basis fields in both bundles. Naturally, we also define $\Omega^0(M, E) := \Gamma(E)$.

Definition 3.13. Let (P, π, M) be a G principal bundle and V a d dimensional vector space. We then call

$$\Omega^k(P, V) := \Gamma\left(V \otimes \Lambda^k T^* P\right) \tag{3.11}$$

the space of **k-forms on P with values in V**. Again, as before, we write

$$\Omega^0(P, V) := \mathcal{C}^\infty(P, V). \tag{3.12}$$

Since, in contrast to before, V is simply a vector space, rather than a bundle, the definition is still to be understood as point wise, though, we only need to choose a single basis on V (and not for every fiber like before).

Definition 3.14. Let $\rho : G \to GL(V)$ be a representation. We then call a form $\omega \in \Omega^k(P, V)$ **ρ-tensorial** if

(i)

$$R_g^*\omega = \rho(g^{-1})\omega \tag{3.13}$$

that is

$$\omega\left(DR_g\left(X_1\right), ..., DR_g\left(X_k\right)\right)|_{R_g(p)} = \rho(g^{-1})\omega\left(X_1, ..., X_k\right)|_p\ ,\quad X_1, ..., X_k \in T_pP \tag{3.14}$$

(ii) if one one of the arguments $X_1, ..., X_k \in T_pP$ is vertical, then $\omega(X_1, ..., X_k)|_p = 0$.

Definition 3.15. We denote with

$$\Omega^k\left(P,\rho\right) := \left\{\omega \in \Omega^k\left(P,V\right)\ \mid \omega\ \rho\text{-tensorial}\right\} \tag{3.15}$$

the space of all ρ-tensorial k-forms on P. Furthermore we notice that

$$\Omega^0\left(P,\rho\right) = \left\{\lambda \in \Omega^0\left(P,V\right)\ \mid \lambda\left(R_g(p)\right) = \rho(g^{-1})\lambda(p)\right\}. \tag{3.16}$$

As the latter remark alludes to, it is possible to generalize theorem (3.11) for all k.

Theorem 3.16. *There is a one to one correspondence between the set of $(P\times_\rho V)$ valued k-forms on M, $\Omega^k\left(M, P\times_\rho V\right)$ and the space of ρ-tensorial k-forms on P, $\Omega^k\left(P,\rho\right)$. That is*

$$\Omega^k\left(M, P\times_\rho V\right) \cong \Omega^k\left(P,\rho\right). \tag{3.17}$$

3.2 Connections on Principal Bundles

Definition 3.17. Let (P,π,M) be a principal G bundle. For an element $X\in\mathfrak{g}$ in the Lie algebra of g we define the **through X defined vector field on P** via

$$\tilde{X}|_p := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} p\exp\left(tX\right)\ ,\quad p\in P \tag{3.18}$$

where we understand $p\exp\left(tX\right) \equiv R_{\exp\left(tX\right)}(p)$ as the curve through $p\in P$ generated by the right action of the subgroup $\exp\left(tX\right)$. Hence, $\tilde{X}\in\mathfrak{X}\left(P\right)$. It is easy to see that such generated vector fields are always vertical. Furthermore the mapping $X\in\mathfrak{g}\mapsto\tilde{X}|_p\in T_p\mathrm{Vert}$ is a linear isomorphism $\forall p\in P$.

Definition 3.18. Let (P,π,M) be a principal G-bundle. A **connection on** P is a $\mathfrak{g}$ valued 1-form $A\in\Omega^1\left(P,\mathfrak{g}\right)$ with

(i) $R_g^*A = \mathrm{Ad}\left(g^{-1}\right)A\ ,\quad \forall g\in G$

(ii) $A(\tilde{X})|_p = X\ ,\quad \forall X\in\mathfrak{g}$ and generated $\tilde{X}\in\mathfrak{X}\left(P\right)$.

Example 3.19.

- On the trivial bundle $P = M\times G$, the Maurer-Cartan form of G (viewed as a vertical form on P) is a connection.
- On a homogeneous bundle $G\to G/H$, both the Maurer-Cartan form of G as well as that of H (defined on G) are connections.

Definition 3.20. Let (P,π,M) be a principal G-bundle and A a connection. At $p\in P$ we define

$$T_p^A\mathrm{Hor} := \ker\left(A|_p\right) \subset T_pP \tag{3.19}$$

as the **horizontal subspace with respect to** A at the point p. Further more a vector field $X\in\mathfrak{X}\left(P\right)$ is called horizontal (wrt. A) if it is horizontal at each point, i.e. if $A(X)\equiv 0$.

One can show that (i) right translation maps A-horizontal spaces into each other bijectively and that (ii) each tangent space decomposes into a direct sum between the vertical subspace and the A-horizontal one. Indeed, strictly speaking a *connection* on a principal bundle is such a choice of horizontal subspaces and it is a theorem that such a choice always coincides with the choice of a *connection 1-form* as we have defined it. From now on we denote with $\mathcal{C}\left(P\right)\subset\Omega^1\left(P,\mathfrak{g}\right)$ the space of all connections on a given principal G-bundle (P,π,M).

Theorem 3.21. *The space of all connections $\mathcal{C}(P)$ is an $\Omega^1(P, \mathrm{Ad})$ affine space. That is,*

$$(A_1 - A_2) \in \Omega^1(P, \mathrm{Ad}) \quad \forall A_{1,2} \in \mathcal{C}(P) \tag{3.20}$$

the difference of two connections is an Ad-tensorial 1-form.

Definition 3.22. Let $\sigma_i : U_i \to P$ be a family of sections where $\{U_i\} \subset M$ cover M and $A \in \mathcal{C}(P)$. We then call

$$A_i := \sigma_i^* A \in \Omega^1(U_i, \mathfrak{g}) \tag{3.21}$$

local connection (1-form) of A with respect to σ_i.

An important consequence of the transitivity of the G action on the fibers of a principle bundle is that for two local sections σ_i and σ_j with overlapping pre-images $U_i \cap U_j \neq \emptyset$, we can always find a $\alpha_{ij} : U_i \cap U_j \to G$ such that

$$\sigma_j(x) = \sigma_i(x)\alpha_{ij}(x). \tag{3.22}$$

We call those functions *transition functions* of the sections. With the use of transition functions we get two powerful results.

Theorem 3.23.

(i) Let $A \in \mathcal{C}(P)$ and $\{A_i\}$ the local ones with respect to the sections $\{\sigma_i\}$. Furthermore, let $\alpha_{ij} : U_i \cap U_j \to G$ be the transition functions. Then, different local connections are related via

$$A_j = \mathrm{Ad}\left(\alpha_{ij}^{-1}\right) A_i + \alpha_{ij}^{-1} \mathrm{d}\alpha_{ij}. \tag{3.23}$$

(ii) If $\{A_i\}$ is a family of local connections who obey the above relations, then there exists a unique $A \in \mathcal{C}(P)$ such that $A_i = \sigma_i^* A$.

The theorem establishes local connections as a class of geometrical objects defined on the base space of a principle bundle (or any manifold for that matter), which are characterized by their transformation behavior (3.23). Indeed, the transformation is nothing new, as it is precisely the same as for Christoffel symbols which indeed are a special case of local connections.

Definition 3.24. Let $A \in \mathcal{C}(P)$. Let $\mathrm{pr}^A : T_pP \to T_p^A\mathrm{Hor}$ be the A **horizontal projection**. We can express it with A via

$$\mathrm{pr}^A(v) = v - \widetilde{A(v)}|_p \in T_p^A\mathrm{Hor}\ , \quad v \in T_pP. \tag{3.24}$$

Definition 3.25. Let $A \in \mathcal{C}(P)$. The **exterior covariant derivative with respect to** A is defined as

$$\mathrm{d}^A \ : \Omega^k(P, V) \to \Omega^{k+1}(P, V) \tag{3.25}$$

$$\mathrm{d}^A\omega(v_0, ..., v_k) := \mathrm{d}\omega(\mathrm{pr}^A(v_0), ..., \mathrm{pr}^A(v_k))\ , \quad v_i \in T_pP \tag{3.26}$$

or in short

$$\mathrm{d}^A = \mathrm{pr}^A \circ \mathrm{d}. \tag{3.27}$$

Notice that we have defined it for any vector valued form.

Before proceeding, we extend the notion of the wedge product to vector and endomorphism valued forms, which we have used implicitly up until now.

Definition 3.26. Let $\omega_{1,2} \in \Omega^{k_{1,2}}(P, \mathrm{End}(V))$ and $\alpha \in \Omega^k(P, V)$. We can thus expand them

$$\omega_{1,2} = T_{i_{1,2}} e^{i_{1,2}}\ , \ \alpha = v_i e^i \tag{3.28}$$

for $T_{i_{1,2}} \in \mathrm{End}(V)$, $v_i \in V$ and $e_{i_{1,2}} \in \Omega^{k_{1,2}}(P)$, $e_i \in \Omega^k(P)$. We then define the wedge product of endomorphism valued forms with each other and with vector valued forms as

$$\omega_1 \wedge \omega_2 := T_{i_1} \circ T_{i_2} e^{i_1} \wedge e^{i_1} \in \Omega^{k_1+k_2}(P, \mathrm{End}(V)) \tag{3.29}$$

and

$$\omega_1 \wedge \alpha := T_{i_1}(v_i) e^{i_1} \wedge e^i \in \Omega^{k_1+k}(P, V). \tag{3.30}$$

Furthermore, let $\rho : G \to \mathrm{GL}(V)$ be a representation and denote with $\rho_* := \rho_*|_e \equiv \mathrm{D}\rho|_e$. Hence, ρ_* is a Lie algebra homomorphismp between $\mathfrak{g}$ and $\mathrm{Im}(\rho_*) \subset \mathfrak{gl}(V) = \mathrm{End}(V)$.

Theorem 3.27. *Let $A \in \mathcal{C}(P)$. Furthermore, let $\rho : G \to \mathrm{GL}(V)$ be a representation and denote with $\rho_* := \rho_*|_e \equiv \mathrm{D}\rho|_e$. Hence, ρ_* is a Lie algebra homomorphismp between $\mathfrak{g}$ and $\mathrm{Im}(\rho_*) \subset \mathfrak{gl}(V) = \mathrm{End}(V)$. Then:*

(i) For $\alpha \in \Omega^k(P, V)$ with $R_g^* \alpha = \rho\left(g^{-1}\right)\alpha$ it follows that

$$\mathrm{d}^A \alpha \in \Omega^{k+1}(P, \rho). \tag{3.31}$$

(ii) For $\alpha \in \Omega^k(P, \rho)$ it follows that

$$\mathrm{d}^A \alpha = \mathrm{d}\alpha + \rho_*(A) \wedge \alpha. \tag{3.32}$$

Especially for the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ we have

$$[\alpha, \beta] := \mathrm{ad}(\alpha) \wedge \beta = [\alpha_i, \beta_j] e^i \wedge e^j \ , \quad \alpha\,, \beta \in \Omega^{k_\alpha, k_\beta}(P, \mathfrak{g}). \tag{3.33}$$

With this, we are now in the place to define the next important object.

Definition 3.28. Let $A \in \mathcal{C}(P)$, we then call

$$F^A := \mathrm{d}^A A \in \Omega^2(P, \mathrm{Ad}) \tag{3.34}$$

the **curvature**-form or **field strength** of A.

Theorem 3.29. *Let $A \in \mathcal{C}(P)$, then*

$$F^A = \mathrm{d}A + \frac{1}{2}[A, A] \ , \quad \textit{Cartan Structure equation} \tag{3.35}$$

$$\mathrm{d}^A F^A = 0 \ , \quad \textit{Bianchi identity.} \tag{3.36}$$

We see that the relations are in total analogy to the curvature tensor and Christoffel symbols in ordinary differential geometry. Indeed, our principal bundle formalism is a generalization of the former.

In physics we view the connections and curvatures as gauge potentials and their field strengths.[8] Since the field strength is a Lie algebra valued two-form, it has components

$$F_{ij} = F^a_{ij} I_a \tag{3.37}$$

where $\{I_a\}$ is some basis on the Lie algebra. Hence, on a four dimensional (Lorentzian) base space, we define for each direction in the Lie algebra the *color-electric- and magnetic fields* E^a_i, B^a_i as

$$F^a_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_3 & 0 \end{pmatrix}^a \tag{3.38}$$

that is

$$E^a_i := F^a_{0i} \tag{3.39}$$

$$B^a_i := \frac{1}{2}\varepsilon_{ijk} F^a_{jk}. \tag{3.40}$$

Definition 3.30. Let (P, π, M) be a principal G-bundle. We call an equivariant diffeomorphism $\varphi : P \to P$, that is

$$\varphi(pg) = \varphi(p) g \tag{3.41}$$

a **transformation of** P. If a transformation φ additionally fixes the fibers, i.e.

$$\pi \circ \varphi = \pi \tag{3.42}$$

we call it a **gauge transformation**.

[8] That is, the local fields on the base space when pulled back along some section.

Theorem 3.31.

(i) The pullback of a connection along a transformation is again a connection.

(ii) The set of gauge transformation is an infinite dimensional Lie subgroup of the set Diff(P)*. Furthermore, gauge transformations map sections into sections via composition in a manner that (3.23) is preserved. We call the group of gauge transformations gauge group.*

(iii) The group of gauge transformations is isomorphic to the set of maps $\mathcal{G}(P) := \{\alpha : P \to G \mid \alpha(pg) = g^{-1}\alpha(p)g\}$, hence we will always mean such maps, when we talk about gauge transformations.

Theorem 3.32. *Let $A \in \mathcal{C}(P)$ and $\alpha : P \to G$ a gauge transformation. Then connection and curvature transform as*

$$\alpha^* A = \mathrm{Ad}(\alpha^{-1})A + \alpha^{-1}\mathrm{d}\alpha \tag{3.43}$$

$$F^{\alpha^* A} = \mathrm{Ad}(\alpha^{-1})F^A. \tag{3.44}$$

3.3 Yang-Mills Theory

After having established the bare mathematical formalism of principal bundles and gauge transformations, we now turn to the physical interpretation of this mathematical apparatus. Many physical theories or models can be (but must not always be) described using connections and curvatures of principal bundles over spacetime. One of the advantages here is that theories or models can be easily constructed purely by guiding oneself on symmetry principles. The price one pays by virtue of invariance, is the introduction of unphysical degrees of freedom which one has to identify or get rid off. Depending on the situation, one can use different action functionals to define the dynamics of the gauge field. One of the most important examples of them is the *Yang–Mills action* which appears in many physical theories and which will be the topic of this section.

In general we will from now on work on a principal G-bundle (P, π, M) over an oriented, n-dimensional, (pseudo-)Riemannian manifold without boundary (M, g), where the structure group G is semisimple. Also denote by $*$ the Hodge star operator on M and by $\langle , \rangle_{\mathfrak{g}}$ an Ad-invariant, non-degenerate inner product on $\mathfrak{g}$. An important remark on the latter is that, conventionally this product is assumed to be positive definite and thus one often additionally assumes that the structure group G is compact, since in that case the existence of such a product is guaranteed by the Killing-form. As this assumption is not needed for all results and since we will be interested in the non-compact case later on, we only assume G to be semisimple unless stated otherwise. We also remark that the usage of the identification theorem 3.16 is understood implicitly.

Definition 3.33. For $\alpha, \beta \in \Omega^k(M, \mathrm{Ad}P)$ we define the inner product as

$$\langle \alpha, \beta \rangle := \sum_{1 \le i_1 < \ldots < i_k \le n} \langle \alpha(E_{i_1}, ..., E_{i_k}), \beta(E_{i_1}, ..., E_{i_k}) \rangle_{\mathfrak{g}} \tag{3.45}$$

where $\{E_i\} \subset \mathfrak{X}(M)$ is an orthonormal basis. In particular for pure tensor products $\alpha = I_A \otimes \tilde{\alpha}^A, \beta = I_A \otimes \tilde{\beta}^A$ this means

$$\langle \alpha, \beta \rangle = \langle I_A, I_B \rangle_{\mathfrak{g}} \langle \tilde{\alpha}^A, \tilde{\beta}^B \rangle \tag{3.46}$$

where the second inner product is the usual one induced by g. Furthermore for $\omega_{1,2} = I_A \otimes \tilde{\omega}^A_{1,2} \in \Omega^{k_1,k_2}(M, \mathrm{Ad}P)$ we define the map

$$\langle \omega_1 \wedge \omega_2 \rangle := \langle I_A, I_B \rangle_{\mathfrak{g}} \tilde{\omega}^A_1 \wedge \tilde{\omega}^B_2 \in \Omega^{k_1+k_2}(M) \tag{3.47}$$

which satisfies for $k_1 = k_2 = k$

$$\langle \omega_1 \wedge *\omega_2 \rangle = \langle \omega_1, \omega_2 \rangle \, \mathrm{dVol}. \tag{3.48}$$

In particular we see that the star pulls through the tensor product.

The newly defined wedge product has a particularly interesting relation to the exterior. covariant derivative. For $\omega_{1,2} \in \Omega^k(M, \mathrm{Ad}P)$ and $A \in \mathcal{C}(P)$ we have

$$\mathrm{d}\,\langle \omega_1 \wedge \omega_2 \rangle = \langle \mathrm{d}^A \omega_1 \wedge \omega_2 \rangle + (-1)^k \langle \omega_1 \wedge \mathrm{d}^A \omega_2 \rangle \,. \tag{3.49}$$

From this relation we are able to define the *adjoint* of the exterior covariant derivative

Theorem 3.34. *The Operator* $\delta^A : \Omega^{k+1}(M, \mathrm{Ad}M) \to \Omega^k(M, \mathrm{Ad}M)$ *defined as*

$$\delta^A := (-1)^{mk+1} * \mathrm{d}^A * \tag{3.50}$$

is the adjoint of d^A*, that is*

$$\int_M \langle \mathrm{d}^A \alpha, \beta \rangle \,\mathrm{dVol} = \int_M \langle \alpha, \delta^A \beta \rangle \,\mathrm{dVol} \tag{3.51}$$

we call δ^A *the **exterior covariant codifferendial**. Note however that this result relies on the fact that* M *has no boundary.*

Definition 3.35. The **Yang–Mills functional** also called **Yang–Mills action** $S : \mathcal{C}(P) \to \mathbb{R}$ is defined as

$$S[A] := \int_M \langle F^A \wedge *F^A \rangle \tag{3.52}$$

where F^A is to be understood to be the identified form in $\Omega^2(M, \mathrm{Ad}P)$.

Theorem 3.36.

(i) The equations of motion derived by varying the Yang-Mills action are

$$\delta^A F^A = 0 \;\Leftrightarrow\; \mathrm{d}^A * F^A = 0 \tag{3.53}$$

*and are called the **Yang–Mills equations**, which are in general a system of non-linear, second order PDE's. A solution to the Yang-Mills equations is called **Yang–Mills connection**.*

(ii) From the Ad*-invariance it is readily seen that the Yang–Mills action is invariant under gauge transformations.*

(iii) An immediate consequence of the latter is that the gauge transformation of a Yang–Mills connection is yet again a Yang-Mills connection.

In physics it is often usual practise to just use the Killing form as the inner product on $\mathfrak{g}$, though strictly speaking we only need the Ad-invariance. Furthermore there are different normalizations used for $S[A]$; Usually a factor of $\frac{1}{4}$ or $\frac{1}{4\alpha}$ is included where $\alpha = g^2 \in \mathbb{R}$ is a coupling constant. Also, there is the possibility to couple matter to a gauge field, if this is *not* done, like for us, one speaks of *pure* Yang–Mills theory. We will not delve into the different mechanisms of matter coupling as we will only work with pure Yang–Mills. It is none the less noteworthy that the most prominent and familiar way of doing this is via *minimal coupling* where one simply replaces all derivatives with gauge-covariant ones, for a field lying in some associated bundle.

Prominent examples of Yang–Mills theories describing physical systems are electromagnetism $U(1)$, the strong interaction $SU(3)$, the electroweak interaction $SU(2) \times U(1)$ or, combined, the standard model of particle physics $SU(3) \times SU(2) \times U(1)$. Especially when doing physics, one should not confuse gauge symmetry as a symmetry in the usual sense (Noethers theorem). Since the physical fields reside in the field strength F[9], the degrees of freedom provided by a gauge field A *are not all physical* which is reflected in the gauge *redundancy* (as one may call it). Hence, what one really is interested in is not the Yang–Mills solutions on $\mathcal{C}(P)$ but rather on $\mathcal{C}(P)/\mathcal{G}(P)$. Having gauge symmetry present allows one to (partially) remove it, i.e. restricting oneself to physical degrees of freedom, by *fixing a gauge.* Fixing a gauge means to impose additional equations on the gauge field such that each solution lies in a different gauge orbit. It is not clear how to do this for a general non-abelian gauge theory.

[9]For non-abelian theories this statement has a few caveats, which we will not delve into.

Theorem 3.37. *In four dimensions, the Yang–Mills action is conformally invariant, we have*

$$S = \int \langle F \wedge *F \rangle = \frac{1}{2} \int \langle F_{\mu\nu}, F_{\alpha\beta} \rangle_{\mathfrak{g}} \, g^{\mu\alpha} g^{\nu\beta} \sqrt{-g} \mathrm{d}^4 x \tag{3.54}$$

which for $g \mapsto \varphi(x) g$ maps to itself since we have in total two powers of g^{-1} and two of g.

Theorem 3.38. *For $n = 4$ a connection $A \in \mathcal{C}(P)$ whose curvature satisfies*

$$*F = \pm F \tag{3.55}$$

is called ***self- and anti-self dual*** *and automatically solves the Yang–Mills equations due to the Bianchi identity. That means that the (anti-)self duality condition is a first integral of the Yang–Mills equations. In physics we also call such solutions* ***instantons****. Furthermore the quantity*

$$\int_M \langle F \wedge F \rangle = \int_M \langle F, *F \rangle \, \mathrm{dVol} \tag{3.56}$$

is independent of the choice of A and thus is a topological invariant of the principal bundle, called the ***instanton number****. When G is compact, the instanton number can be used to classify principal bundles and the existence of instantons on them. Finally, if G is compact, then any instanton solution is an absolute minimum of the Yang-Mills action.*

4 Coset space dimensional reduction

The Yang–Mills equations by themselves are quite complicated, especially in the non-abelian case. Hence, further restrictions on the bundle structure or the solution itself are imperative to make the problem more approachable. One way of drastically simplifying the equations is given by the 'coset space dimensional reduction' (CSDR) scheme [1], [21], [22], whose introduction is the topic of this section. In general, the idea of CSDR is to consider a spacetime $M \cong S \times G/H$ which can be decomposed into a manifold S and coset space G/H. Then, the big group G is chosen to be the structure group and invariance under the G action on the spacetime is imposed on the Yang–Mills field (up to gauge transformations), hence yielding a subclass of symmetric solutions whose equations of motion become easier to solve.[10] In the special case we are interested in, we make the spacetimes into cylinders over the coset, that is, $S = \mathbb{R}$. Notice that the G-invariance implies that the equations will be independent of the coset coordinates. Hence, in the cylindrical case the Yang–Mills system will reduce to a system of ODE's for the remaining degrees of freedom.

We start off with the coset geometry as outlined in chapter 2.4. Let G be a connected and semisimple Lie group and $H \subset G$ a Lie subgroup. The coset G/H then will be a reductive homogeneous space with respect to the Killing-orthogonal decomposition of the Lie algebra

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \tag{4.1}$$

where $\mathfrak{h}$ is the Lie algebra of H. Then, as before, denote with $\{I_A\} \subset \mathfrak{g}$ an Killing-orthogonal basis and split it up into bases $\{I_\alpha\} \subset \mathfrak{h}$ and $\{I_a\} \subset \mathfrak{m}$. And, again, reductivity then also splits the commutation relations into

$$[I_\alpha, I_\beta] = f^{\gamma}_{\ \alpha\beta}\, I_\gamma\ , \quad [I_\alpha, I_a] = f^{b}_{\ \alpha a}\, I_b\ , \quad [I_a, I_b] = f^{c}_{\ ab}\, I_c + f^{\alpha}_{\ ab}\, I_\alpha. \tag{4.2}$$

We choose the basis in such a way that the magnitude of the Killing-square is the same for all generators. That is, the components of the Killing-form read

$$K_{AB} = K(I_A, I_B) = \mathcal{D}\, \tilde{\eta}_{AB} \tag{4.3}$$

with $\mathcal{D} \in \mathbb{R}$ and $\tilde{\eta}_{AB} = \eta^{(p,q)}_{AB}$ where (p,q) is the signature of the Killing-form. Now let $\{\hat{e}^A\} \subset \Omega^1(G)$ be the left-invariant dual basis of the generators. By contraction of these forms with the components of the the Killing-form, we obtain the Cartan-Killing metric on G as

$$g := \mathcal{D}\, \tilde{\eta}_{AB}\, \hat{e}^A \otimes \hat{e}^B \tag{4.4}$$

Moving on, we choose some section of the coset $\sigma : G/H \to G$ and pull back the forms as

$$\{e^A\} := \{\sigma^* \hat{e}^A\} \subset \Omega^1(G/H) \tag{4.5}$$

which will then decompose into a basis $\{e^a\}$ and the rest, which will be a linear combination of the former $e^\alpha = \chi^\alpha_{\ b}\, e^b$, with some functions $\chi^\alpha_{\ b} \in \Omega^0(G/H)$. We then equip the coset with the normalized pull back of the coset part of the Cartan-Killing metric, that is

$$g_{G/H} := \tilde{\eta}_{ab}\, e^a \otimes e^b =: \tilde{\eta} \tag{4.6}$$

which is G-left invariant. Now, we consider a cylinder over our coset, which will be the base space M of our principal bundle. We have

$$M := \mathbb{R} \times G/H \tag{4.7}$$

together with the trivial product metric

$$g := \mathrm{d}u \otimes \mathrm{d}u + \tilde{\eta} =: g_{\mu\nu}\, e^\mu \otimes e^\nu \tag{4.8}$$

where $e^0 := \mathrm{d}u$. Hence, the components of the metric in the basis $\{e^0, e^a\}$ read

$$g_{\mu\nu} = \begin{pmatrix} 1 & \\ & \tilde{\eta}_{ab} \end{pmatrix}. \tag{4.9}$$

[10] Of course, there is also the possibility of choosing a different gauge group, though this will not be of concern to our endeavor.

Notice that the signature of our cylinder thus becomes $(p+1,q)$. Later on we want the signature to be Lorentzian, for which we will need to tune certain signs appropriately. Together with M we now consider the (trivial) principal G bundle $(G\times M,\pi,M)$ to do Yang–Mills theory.

We now want to require that our gauge fields are symmetric (up to gauge transformations) under the natural left G action on M. Any gauge field can be expanded in our of one forms above, we have

$$A = A_0\, e^0 + A_\alpha\, e^\alpha + A_a\, e^a. \tag{4.10}$$

Notice that we include a part with the redundant forms $\{e^\alpha\}$. We will shortly see that this is well adapted to the symmetry requirement. Furthermore we will work in the 'temporal gauge' and thus set $A_0 \equiv 0$, i.e. the gauge field has no part in the foliation direction. Now, to obtain the ansatz which obeys the constraint we utilize a theorem [21].

Theorem 4.1. *Let (P,π,M) be a principal G bundle and $K\subset \mathrm{Diff}(P)$ a Lie subgroup of automorphisms of P. Further fix $p_0\in P$ and let $J := \{j\in K \mid \pi(jp_0)=\pi(p_0)\}\subset K$ be the Lie subgroup of K which fixes the base point of p_0. Then there exists a Lie group homomorphism $\lambda: J\to G$ such that $jp_0 = p_0\lambda(j)\ \forall j\in J$. Now also denote with λ the induced Lie algebra homomorphism $\lambda:\mathfrak{j}\to\mathfrak{g}$. Also let K be reductive with respect to the decomposition $\mathfrak{k}=\mathfrak{j}\oplus\mathfrak{m}$. Then there is a one to one correspondence between K-invariant connections $\omega\in\mathcal{C}(P)$ and the set of linear maps $\Lambda_{\mathfrak{m}}:\mathfrak{m}\to\mathfrak{g}$ with*

$$\Lambda_{\mathfrak{m}}(\mathrm{Ad}(j)(X)) = \mathrm{Ad}(\lambda(j))(\Lambda_{\mathfrak{m}}(X)). \tag{4.11}$$

The connections are then given by

$$\omega|_{p_0}(\tilde X) = \Lambda(X)\ ,\ \ \tilde X|_{p_0} = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\exp(sX)p_0\, , X\in\mathfrak{k} \tag{4.12}$$

where

$$\Lambda(X) := \begin{cases}\lambda(X)\ ,\ X\in\mathfrak{j}\\ \Lambda_{\mathfrak{m}}(X)\ ,\ X\in\mathfrak{m}.\end{cases} \tag{4.13}$$

To apply this theorem to our setting we first observe that for us $K=G$ is acting via $p_0=(g_0,x_0)\mapsto (g_0,k\,x_0)$, i.e. $J\cong \mathrm{Stab}(x_0)\cong H$. Thus we can choose λ to be the natural inclusions of H and $\mathfrak{h}$ respectively, that is, $\lambda=\mathrm{id}_{H,\mathfrak{h}}$. Reductivity of $K=G$ was assumed in our setup and since $\mathfrak{j}=\mathfrak{h}$, the splitting in the theorem is just the Killing-orthogonal splitting from our setup, that is $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$. Considering now our expansion

$$A = A_\alpha\, e^\alpha + A_a\, e^a. \tag{4.14}$$

We can apply the correspondence of the theorem to determine the components of the gauge field. First we have that $\Lambda(X)=X\ \forall X\in\mathfrak{h}$, since together with (4.12) this simply amounts to the canonical Maurer-Cartan form of H we get

$$A_\alpha = \Lambda(I_\alpha) = I_\alpha. \tag{4.15}$$

For the second part in (4.12) we first expand the map $\Lambda_{\mathfrak{m}}$ in our basis

$$A_a = \Lambda(I_a) = \Lambda_{\mathfrak{m}}(I_a) =: X^b_{\ a}\, I_b + X^\beta_{\ a}\, I_\beta. \tag{4.16}$$

Together with (4.11), reductivity of the coset now readily implies that the components $X^\beta_{\ a}\equiv 0$ vanish. The gauge potential now reads

$$A = I_\alpha\, e^\alpha + X^b_{\ a}\, I_b\, e^a. \tag{4.17}$$

It is thus a deviation of the canonical H-Maurer-Cartan form. What is left is the constraint in (4.11) which (infinitesimally) reads

$$X^c_{\ b}\, f^b_{\ \alpha a} = f^c_{\ \alpha b}\, X^b_{\ a}\ \ \Leftrightarrow\ \ [I_\alpha, X_a] = f^b_{\ \alpha a} X_b, \tag{4.18}$$

that is, the components X_a must lie in the adjoint representation $\mathrm{ad}(\mathfrak{h})|_\mathfrak{m}$ of $\mathfrak{h}$ restricted to the coset generators. To obtain the final form of our symmetric ansatz we now need to solve this condition. For a reductive coset the adjoint representation of the subalgebra decomposes into a direct sum

$$\mathrm{ad}(\mathfrak{h})|_\mathfrak{g} = \mathrm{ad}(\mathfrak{h})|_\mathfrak{h} \oplus \underbrace{\mathrm{ad}(\mathfrak{h})|_\mathfrak{m}}_{=:\mathcal{R}} . \tag{4.19}$$

Where the second part $\mathcal{R}$ may in general be reducible. We can write it as the sum $\mathcal{R} = \bigoplus_{i=1}^{q} \mathcal{R}_i$, where the $\mathcal{R}_i$ are irreps of $\mathfrak{h}$. Now, looking at (4.18), we see that the X_a must transform exactly as the coset generators I_a. The most general solution to this constraint is now to introduce q ´scalar' degrees of freedom $\phi_i(u)$ for each irrep $\mathcal{R}_i$. We do this straight forwardly by first finding the linear transformation $I_a \mapsto T_a^b\, I_b =: \bar{I}_a$ which block-diagonalizes the representation $\mathcal{R}$. That is, the generators $\bar{I}_a$ are arranged into bases $\{\bar{I}_1^{(1)}, ..., \bar{I}^{(1)}_{\dim(\mathcal{R}_1)}, ..., \bar{I}_1^{(q)}, ..., \bar{I}^{(q)}_{\dim(\mathcal{R}_q)}\}$ of each invariant subspace. In this basis it is straight forward to introduce the scalar degrees of freedom for each of the irreps by simply multiplying by the block-diagonal matrix given by

$$\Phi := \begin{pmatrix} \phi_1\, \mathbb{1}_{\dim(\mathcal{R}_1)} & & \\ & \ddots & \\ & & \phi_q\, \mathbb{1}_{\dim(\mathcal{R}_q)} \end{pmatrix} . \tag{4.20}$$

Transforming this then back with T^{-1} yields the components X_a.

$$X_a = X^b_{\ a}\, I_b = \left(T^{-1}\Phi T\, I\right)_a \tag{4.21}$$

Of course the basis change becomes trivial when the $\{I_a\}$ are already adapted to the decomposition of the representation, which is always possible to do from the beginning. Our symmetric ansatz for the gauge field thus reads

$$A = A_a\, e^a = I_\alpha\, e^\alpha + X_a(u)\, e^a, \tag{4.22}$$

that is,

$$A_a(u) = \chi^{\alpha}_{\ a}\, I_\alpha + X_a(u). \tag{4.23}$$

Hence, we have reduced the system to a set of scalar degrees of freedom $\phi_i(u)$, which only depend on the foliation parameter and whose number is determined by the representation theory of our coset. Having our symmetric ansatz, we now want to obtain the equations of motion. First we calculate the field strength. We start by expanding it in our basis of one forms

$$F = \mathrm{d}A + A \wedge A = F_{0a}\, e^0 \wedge e^a + \frac{1}{2} F_{ab}\, e^a \wedge e^b. \tag{4.24}$$

Plugging in our symmetric ansatz we obtain after straight forward calculation the components as

$$F_{0a} = \dot{X}_a \ , \quad F_{ab} = -f^c_{\ ab} X_c - [I_a, I_b] + [X_a, X_b] \tag{4.25}$$

where the dot denotes differentiation with respect to the foliation parameter u. At this stage we now have two routes to proceed. We can either first vary the Yang–Mills action and plug in our symmetrized ansatz into the resulting equation (which of course will be just the Yang–Mills equations). Or we first symmetrize the action and then vary the reduced action with respect to our reduced degrees of freedom ϕ_i. We have the following picture.

$$\begin{array}{ccc} S[A] & \xrightarrow{\text{symmetrize}} & S_{\text{red}}[\phi] \\ \big\downarrow \delta & & \big\downarrow \delta \\ \mathrm{d}^A * F & \xrightarrow{\text{symmetrize}} & \text{e.o.m. for } \{\phi_i\} \end{array}$$

It is indeed in general *not* true that these routs yield the same equations for the ϕ_i. Statements about when both routs are equivalent, i.e. the diagram commutes, are captured in the principle of symmetric criticality [23]. Most of the time - and certainly in our cases - the groups are well behaved enough for the principle to hold. Hence, we will later on only work with the reduced action and reduced

Lagrangian respectively, which will have the advantage of making many calculations easier. Though, for completeness' sake we will also give the equations obtained by plugging in the symmetric ansatz into the Yang–Mills equations directly. On our cylinder the Yang–Mills equations decompose into two independent sets, we have

$$E_a F^{a0} + \Gamma_{a\ b}^{\ a} F^{b0} + [A_a, F^{a0}] = 0 \tag{4.26}$$

$$E_0 F^{0b} + E_a F^{ab} + \Gamma_{c\ a}^{\ c} F^{ab} + [A_a, F^{ab}] = 0. \tag{4.27}$$

Where the E_μ are the dual fields to the e^μ. After plugging in the ansatz these then become

$$[X_a, \dot{X}^a] = 0 \tag{4.28}$$

and

$$\ddot{X}_b - \left(\frac{1}{2} f_{bcd} f_{acd} - f_{bc\alpha} f_{ac\alpha}\right) X_a + \frac{3}{2} f_{bcd} [X_c, X_d] + [X_a, [X^a, X_b]] = 0. \tag{4.29}$$

The constraint (4.28) can always be fulfilled by choosing as the basis $\{I_a\}$ such that $\mathcal{R}$ is already block diagonal, since then $\dot{X}_a \sim X_a$. Thus the Yang–Mills system is reduced to a set of second order, non-linear ordinary matrix differential equations or respectively after obtaining the exact form of the X_a a set of non-linear ODE's for the ϕ_i. With this we conclude our outline of CSDR on cylinders.

5 Application of CSDR to non-compact symmetric spaces

We now come to the main endeavor of this thesis. The CSDR scheme over cylinders as outlined in the previous chapter has been applied by my supervisor and collaborators to different kinds cosets, be they symmetric or only reductive, in e.g. [2], [3], [4] and more recently [5]. Certain symmetric solutions in scenarios where they considered cosets over group manifolds can also be mapped to corresponding CSDR schemes [6], [7], [8], [9]. Especially in [2] where the case of $G/H \cong S^n$ was considered, the cylinders were additionally warped so as to make them conformal to de Sitter space dS_n via the well known spacelike spherical slicing[11]. Furthermore in that work the spheres where written as cosets in three different ways, of which only the well known $S^n \cong SO(n+1)/SO(n)$ is symmetric. Considering the latter and dropping the influence of the warping they found that the system reduces to a single Newton like degree of freedom $\phi(t)$ subject to a double well potential. Likewise in [5] the cases of $G/H \cong H^3$, dS_3 were considered which also resulted in the same equation of motion except with inverted double well potential. What we are going to do, in this first part of the endeavor, is to complete the picture by considering the related cases with the three non-compact symmetric spaces hyperbolic space H^n, de Sitter space dS_n and anti-de Sitter space AdS_n as the cosets.

5.1 Geometric setup

We start off by setting up the geometrical preliminaries. Hyperbolic space H^n, de Sitter- and anti-de Sitter space $(A)dS_n$ are all non-compact symmetric spaces and can be written as the quotients of indefinite orthogonal groups.

$$H^n \cong SO(1,n)/SO(n) \tag{5.1}$$
$$dS_n \cong SO(1,n)/SO(1,n-1) \tag{5.2}$$
$$AdS_n \cong SO(2,n-1)/SO(1,n-1) \tag{5.3}$$

Where we understand that we implicitly take the quotients of the connected components of the identity. A detail which is necessary for the principle of symmetric criticality to hold later on. Furthermore notice that since all these spaces are symmetric, we have that the decompositions $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, in addition to being reductive, also have the property that $\mathfrak{m}$ commutes into $\mathfrak{h}$, that is

$$[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}\ , \quad [\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m}\ , \quad [\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}. \tag{5.4}$$

Let us now first characterize the Lie algebras and find their respective Cartan decompositions. We begin with hyperbolic space H^n. In the well known defining matrix representation the Lie algebra of the Lorentz group $SO(1,n)$ takes the form

$$\mathfrak{so}(1,n) =: \mathfrak{g} = \begin{pmatrix} 0 & \vec{v}^T \\ \vec{v} & \text{skew} \end{pmatrix}\ , \quad \vec{v} \in \mathbb{R}^n \tag{5.5}$$

where the lower skew symmetric $n \times n$ block is the rotation algebra.

$$\mathfrak{h} := \begin{pmatrix} 0 & \vec{0}^T \\ \vec{0} & \text{skew} \end{pmatrix} \cong \mathfrak{so}(n) \hookrightarrow \mathfrak{so}(1,n) \tag{5.6}$$

Hence the rest is given by the boosts

$$\mathfrak{m} := \begin{pmatrix} 0 & \vec{v}^T \\ \vec{v} & 0 \end{pmatrix} \cong \mathbb{R}^n \hookrightarrow \mathfrak{so}(1,n) \tag{5.7}$$

From this the dimensions are readily calculated as

$$\dim(\mathfrak{g}) = \dim(G) = n + \binom{n}{2} = \frac{n(n+1)}{2} \tag{5.8}$$
$$\dim(\mathfrak{h}) = \dim(H) = \binom{n}{2} = \frac{n(n-1)}{2} \tag{5.9}$$
$$\dim(\mathfrak{m}) = \dim(G) - \dim(H) = \dim(G/H) = n. \tag{5.10}$$

[11] We will later on see how such a warping affects the equations of motion.

Next on we need to choose a basis $\{I_A\}$ on $\mathfrak{g}$. Naturally we use the usual one defined as

$$I_a := \begin{pmatrix} 0 & \vec{e_a}^T \\ \vec{e_a} & 0 \end{pmatrix} \;,\; I_\alpha \sim \begin{pmatrix} 0 & \vec{0}^T \\ \vec{0} & \vec{e}_{[a} \otimes \vec{e}^T_{b]} \end{pmatrix} \tag{5.11}$$

where we understand the index α to be symbolic, since there is no canonical way to enumerate anti-symmetric matricies for $n \geq 3$. Next is the Killing-from. It turns out that the Killing-form of $\mathfrak{so}(p,q)$ for $p+q \geq 3$ and $p,q \geq 1$ is given by [12]

$$K(X,Y) \equiv \mathrm{tr}_{\mathrm{adj}}(XY) = (p+q-2)\mathrm{tr}_{\mathrm{def}}(XY) \tag{5.12}$$

Thus we have for $n \geq 2$ that

$$K_{AB} = K(I_A, I_B) = (n-1)\mathrm{tr}(I_A I_B). \tag{5.13}$$

The trace is easily evaluated and we obtain as one might have guessed

$$\begin{aligned} K_{AB} &:= \mathrm{tr}_{\mathrm{adj}}(I_A I_B) = \mathcal{D}_n \begin{pmatrix} -\mathbb{1}_{\binom{n}{2}} & \\ & +\mathbb{1}_n \end{pmatrix} \\ &=: \mathcal{D}_n \; (\tilde{\eta}_{\alpha\beta} \oplus \tilde{\eta}_{ab}) \end{aligned} \tag{5.14}$$

where $\mathcal{D}_n := 2(n-1)$.

Next is de Sitter space dS_n. Since de Sitter space only differs from hyperbolic space in that we quotient out a sub-Lorentz group instead of the rotations, we can obtain the results by a mere reordering of the previous consideration. Again, in the defining representation we can identify the Lie subalgebra as

$$\mathfrak{h} := \begin{pmatrix} 0 & 0 & \vec{w}^T \\ 0 & 0 & \vec{0}^T \\ \vec{w} & \vec{0} & \mathrm{skew} \end{pmatrix} \equiv \mathfrak{so}(1,n-1) \hookrightarrow \mathfrak{so}(1,n)\,, \quad \vec{w} \in \mathbb{R}^{n-1}. \tag{5.15}$$

Where skew in $\mathfrak{h}$ denotes the set of $(n-1) \times (n-1)$ skew symmetric matricies $\cong \mathfrak{so}(n-1)$. The complement is then given by

$$\mathfrak{m} := \begin{pmatrix} 0 & t & \vec{0}^T \\ t & 0 & \vec{x}^T \\ \vec{0} & -\vec{x} & 0 \end{pmatrix} \cong \mathbb{R}^n \hookrightarrow \mathfrak{so}(1,n). \tag{5.16}$$

Alternatively and also more abstractly, we could choose $(n-1)$ boosts and complete them with all their commutators. At the end of the day, we can characterize the decomposition as

$$\mathfrak{h} = \mathrm{span}\{I_\alpha\}\,, \text{with } (n-1) \text{ boosts and } \binom{n-1}{2} \text{ rotations} \tag{5.17}$$

$$\mathfrak{m} = \mathrm{span}\{I_a\}\,, \text{with 1 boost and } \binom{n}{2} - \binom{n-1}{2} = (n-1) \text{ rotatations.} \tag{5.18}$$

The dimensions thus work out to be

$$\dim(\mathfrak{h}) = \dim(H) = \frac{n(n-1)}{2} \tag{5.19}$$

$$\dim(\mathfrak{m}) = \dim(G/H) = n \tag{5.20}$$

which are the same as before, as expected. Finally, the Killing form will again be the same but the decomposition will look different. It is readily seen from before that generators which are anti-symmetric in the defining representation (as matricies) have negative Killing-square and those which are symmetric in the defining representation have positive Killing-square. Thus we can easily write down the decomposition of the Killing-form by appropriately reordering the components, we have

$$K_{AB} = \mathcal{D}_n \begin{pmatrix} \mathbb{1}_{n-1} & & & \\ & -\mathbb{1}_{\binom{n-1}{2}} & & \\ & & 1 & \\ & & & -\mathbb{1}_{n-1} \end{pmatrix} =: \mathcal{D}_n \tilde{\eta}_{AB} \tag{5.21}$$

with

$$\tilde{\eta}_{AB} = \tilde{\eta}_{\alpha\beta} \oplus \tilde{\eta}_{ab} = \begin{pmatrix} \mathbb{1}_{n-1} & \\ & -\mathbb{1}_{\binom{n-1}{2}} \end{pmatrix} \oplus \begin{pmatrix} +1 & \\ & -\mathbb{1}_{n-1} \end{pmatrix}. \tag{5.22}$$

Lastly we have anti-de Sitter space AdS_n. Again working with the defining matrix representation, the Lie algebra $\mathfrak{so}(2, n-1)$ can be written as

$$\mathfrak{so}(2, n-1) =: \mathfrak{g} = \begin{pmatrix} 0 & t & \vec{x}^T \\ -t & 0 & \vec{v}^T \\ \vec{x} & \vec{v} & \text{skew} \end{pmatrix} \cong \underbrace{\mathbb{R}^{n-1}}_{\vec{v}} \oplus \underbrace{\mathbb{R}^{n-1}}_{\vec{x}} \oplus \underbrace{\mathfrak{so}(2)}_{t} \oplus \underbrace{\mathfrak{so}(n-1)}_{\text{skew}}. \tag{5.23}$$

The sub-Lorentz algebra in it then becomes

$$\mathfrak{h} = \begin{pmatrix} 0 & 0 & \vec{0}^T \\ 0 & 0 & \vec{v}^T \\ \vec{0} & \vec{v} & \text{skew} \end{pmatrix} \cong \mathfrak{so}(1, n-1) \hookrightarrow \mathfrak{so}(2, n-1) \tag{5.24}$$

and finally the complement reads

$$\mathfrak{m} = \begin{pmatrix} 0 & t & \vec{x}^T \\ -t & 0 & \vec{0}^T \\ \vec{x} & \vec{0} & \mathbb{0} \end{pmatrix} \cong \mathfrak{so}(2) \oplus \mathbb{R}^{n-1}. \tag{5.25}$$

Hence the dimensions work out to be

$$\dim(\mathfrak{g}) = \dim(G) = \frac{(n-1)(n+2)}{2} + 1 \tag{5.26}$$

$$\dim(\mathfrak{h}) = \dim(H) = \frac{n(n-1)}{2} \tag{5.27}$$

$$\dim(\mathfrak{m}) = \dim(G/H) = n. \tag{5.28}$$

We choose the basis $\{I_A\}$ in the same way as before. In this basis we then again apply equation (5.12) to obtain the Killing-form and its decomposition.

$$K_{AB} = \mathcal{D}_n \begin{pmatrix} \mathbb{1}_{n-1} & & & \\ & -\mathbb{1}_{\binom{n-1}{2}} & & \\ & & \mathbb{1}_{n-1} & \\ & & & -1 \end{pmatrix} =: \mathcal{D}_n \, \tilde{\eta}_{AB} \tag{5.29}$$

Where the upper two blocks realize the Killing-form of the sub-Lorentz algebra and the lower two the coset part.

$$\tilde{\eta}_{AB} = \tilde{\eta}_{\alpha\beta} \oplus \tilde{\eta}_{ab} = \begin{pmatrix} \mathbb{1}_{n-1} & \\ & -\mathbb{1}_{\binom{n-1}{2}} \end{pmatrix} \oplus \begin{pmatrix} \mathbb{1}_{n-1} & \\ & -1 \end{pmatrix} \tag{5.30}$$

Having established the respective Cartan decompositions and Killing-forms, we now define the metrics for the three cylinders. Since the cylinders are going to be the spacetimes of our setup, they must carry Lorentzian signature. Hence, we have to choose the 00-component of the metric depending on the coset part of the respective Killing-forms. Furthermore we will introduce an additional over all factor to keep track of the ambiguity arising between the two choices of mostly plus or mostly minus signature. We have,

$$g = \varepsilon_g(\tilde{g}_{00}\, e^0 \otimes e^0 + \tilde{\eta}_{ab}\, e^a \otimes e^b) = \varepsilon_g\, \tilde{g} \tag{5.31}$$

where $\varepsilon_g = \pm 1$ and $\tilde{g}$ is the 'bare metric'. Now looking at the three Killing-forms, we see that $\tilde{\eta}_{ab}$ is positive definite for H^n, Lorentzian mostly minus for dS_n and Lorentzian mostly plus for AdS_n. Hence we get for the respective bare 00-components -1 for H^n, -1 for dS_n and $+1$ for AdS_n. Observe that we see here that, since de Sitter and anti-de Sitter space are already Lorentzian manifolds, the foliation parameter of the cylinders in these cases is *not* timelike. In these cases the spacetimes look like ordinary $(A)dS_n$ with an additional infinite and flat dimension.

5.2 Reduced Lagrangians

With the cylinders set up, we are now in place to derive the equations of motion for the three cases. As mentioned in the beginning, we are taking the connected components of the identity for the big groups of the cosets. Thus all our groups are real, semisimple, connected and analytic which is sufficient for the principle of symmetric criticality to hold [23]. Hence, we will substitute our ansatz into the Yang–Mills action and derive the reduced Lagrangians instead of plugging the ansatz into the Yang–Mills equations directly. First notice that from the block structure of the Lie algebras in the defining representations above, we can easily see that for all three cases the adjoint of the subalgebra restricted to the complement, i.e. the representation $\mathcal{R}$, is always the vector representation[12] of $\mathfrak{so}(n)$ and $\mathfrak{so}(1, n-1)$ respectively, which are irreducible. Hence, all three system will reduce to a single degree of freedom $\phi(u)$ and the respective ansatz reads

$$\mathcal{A} = I_\alpha e^\alpha + \phi(u) I_a e^a. \tag{5.32}$$

From this we obtain the components of the repective field strengths as

$$\mathcal{F}_{0a} = \dot{\phi} I_a \in \mathfrak{m}\,, \quad \mathcal{F}_{ab} = (\phi^2 - 1) f^\alpha_{ab} I_\alpha = (\phi^2 - 1)\,[I_a, I_b] \in \mathfrak{h}. \tag{5.33}$$

Notice that due to the symmetry of the cosets, color electric components lie only in $\mathfrak{m}$ and color magnetic components in $\mathfrak{h}$. This also implies for $n = 3 \leftrightarrow d = 4$ that there are no (anti-)self dual solutions which are also symmetric. Now, it turns out that we can treat all three cases in parallel until we explicitly have to take into account the different signatures of the respective $\tilde{\eta}_{ab}$. Hence, without specifying the case, we start off with the Yang–Mills action

$$S[\mathcal{A}] = \frac{1}{4\alpha} \int_{\mathbb{R}\times G/H} K(\mathcal{F} \wedge *\mathcal{F}) = \frac{1}{8\alpha} \int_{\mathbb{R}\times G/H} K(\mathcal{F}_{\mu\nu}, \mathcal{F}^{\mu\nu})\, \mathrm{dVol} \tag{5.34}$$

where $\alpha = g^2 \in \mathbb{R}_{\geq 0}$ is some coupling constant. Before proceeding further, notice that our ansatz (5.33) is completely independent of the coset coordinates. Thus we can split the integral in the action to a part on $\mathbb{R}$ and a part on the coset, with $\mathrm{dVol} = \mathrm{d}u \wedge \mathrm{dVol}_{G/H}$ we get

$$S[\phi] = \mathrm{Vol}(G/H)\, \frac{1}{8\alpha} \int_{\mathbb{R}} K(\mathcal{F}_{\mu\nu}, \mathcal{F}_{\mu\nu}) \mathrm{d}u. \tag{5.35}$$

The volume of the coset pulls out of the action as an overall factor which will render it infinite for all of our cases, since all three cosets are non-compact. Next we define the reduced Lagrangian as

$$\mathcal{L}(\phi, \dot{\phi}) = \frac{1}{8\alpha} K(\mathcal{F}_{\mu\nu}, \mathcal{F}^{\mu\nu}). \tag{5.36}$$

Before we insert our ansatz, we also introduce an over all factor in the Killing-form

$$K_{AB} := \varepsilon_K\, \mathcal{D}_n\, \tilde{\eta}_{AB} \tag{5.37}$$

with $\varepsilon_K = \pm 1$. This factor is useful for us to introduce since the non-compactness of our structure groups renders the Killing-forms indefinite. This means that both energy and action of our solutions may become negative or even zero in the presence of non-trivial fields, which is one of the reasons one might call theories with non-compact structure groups non-physical. Of course it all depends on the sub-class of solutions at hand, as there is the possibility to restrict oneself to classes without such 'null-solutions'. In any case, having the factor in place aids us in keeping track of these matters, to which we will come back later on. Returning to the Lagrangians, it turns out that until we need the particular signature of $\tilde{\eta}_{ab}$, all cases can be treated in parallel. Plugging in the ansatz (5.33) into the Lagrangian we first get

$$\mathcal{L}(\phi, \dot{\phi}) = \frac{1}{8\alpha} K(\mathcal{F}_{\mu\nu}, \mathcal{F}^{\mu\nu}) \tag{5.38}$$

$$= \frac{1}{8\alpha} \left(2K(\mathcal{F}_{0a}, \mathcal{F}_{0b}) g^{00} g^{ab} + K(\mathcal{F}_{ma}, \mathcal{F}_{nb}) g^{mn} g^{ab} \right) \tag{5.39}$$

$$= \frac{1}{8\alpha} \mathcal{D}_n \varepsilon_K \, \left(2\tilde{\eta}(\mathcal{F}_{0a}, \mathcal{F}_{0b}) \tilde{g}^{00} \tilde{\eta}^{ab} + \tilde{\eta}(\mathcal{F}_{ma}, \mathcal{F}_{nb}) \tilde{\eta}^{mn} \tilde{\eta}^{ab} \right) \tag{5.40}$$

$$= \frac{1}{8\alpha} \mathcal{D}_n \varepsilon_K \, \left(2\dot{\phi}^2 n \tilde{g}^{00} + (\phi^2 - 1)^2 \tilde{\eta}([I_a, I_b], [I_a, I_b]) \tilde{\eta}^{aa} \tilde{\eta}^{bb} \right) \tag{5.41}$$

$$= \frac{1}{2\alpha} \mathcal{D}_n n \varepsilon_K \tilde{g}^{00} \, \left(\frac{1}{2}\dot{\phi}^2 + \frac{\tilde{g}^{00}}{4n} \mathcal{S}_n (\phi^2 - 1)^2 \right). \tag{5.42}$$

[12] i.e. again the defining representation.

Where we used that $\tilde{\eta}_{ab}\tilde{\eta}^{ab} = \dim(\mathfrak{m}) = n$. Naturally, the over all sign of the metric drops out. Thus, without further specifying the Killing-form, regardless of the case, the behavior of the remaining degree of freedom will be like that of an one dimensional Newtonian particle subject to a double well- or inverted double well potential

$$V(\phi) = -\frac{\tilde{g}_{00}}{4n}\mathcal{S}_n(\phi^2 - 1)^2 \tag{5.43}$$

where we abbreviated

$$\mathcal{S}_n := \tilde{\eta}([I_a, I_b], [I_a, I_b])\tilde{\eta}^{aa}\tilde{\eta}^{bb} = \sum_{I,J\in\mathfrak{m}} \|[I,J]\|^2_{\tilde{\eta}}\, \|I\|^2_{\tilde{\eta}}\, \|J\|^2_{\tilde{\eta}}. \tag{5.44}$$

Hence, to obtain the Lagrangians for all three cases, we simply have to fix the corresponding Killing-froms, which in turn fixes $\tilde{g}_{00}$ and evaluate the double sum above. The evaluation of the double sum can be done in different ways, the most straight forward of which would be to utilize the structure constants and simply calculating the result. The drawback of that approach though is that, depending on the structure constants, if one even has a closed expression for them that is, it can be cumbersome to deal with all the indices appearing. We will thus go with another approach and resort to simple combinatoric arguments to obtain the $\mathcal{S}_n$.

We begin with hyperbolic space H^n. This case is straight forward as $\tilde{\eta}_{AB}$ decomposes in such a way that it is positive definite on $\mathfrak{m}$ and negative definite on $\mathfrak{h}$. Because of the positve definiteness on the complement we get that $\tilde{g}_{00} = -1$ is fixed. For the double sum $\mathcal{S}_n$ first notice that the summand is the product of the $\tilde{\eta}$-squares of two elements in $\mathfrak{m}$ multiplied with the $\tilde{\eta}$-square of their commutator. Since the coset is symmetric, the commutator always lands in $\mathfrak{h}$. Furthermore for fixed index a we have that the kernel of the map $\mathrm{ad}(I_a) = [I_a, \cdot] : \{I_a\} \subset \mathfrak{m} \to \{I_\alpha\} \subset \mathfrak{h}$ just consists of I_a itself. Hence, the summand is always -1 and 0 if and only if $I = J$. Finally, realizing that we are summing twice over an index range of n and taking into account that the commutator is anti-symmetric, we readily infer that $\mathcal{S}_n = -2\binom{n}{2}$. Returning to the general expression for the potential (5.43) we obtain for the hyperbolic case

$$V_{H^n}(\phi) = -\frac{1}{8}\mathcal{D}_n(\phi^2 - 1)^2 \tag{5.45}$$

where we have used that $\frac{1}{2n}\binom{n}{2} = \frac{1}{8}2(n-1)$. It is thus an inverted double well.

Next is de Sitter space dS_n. Here the decomposition splits $\tilde{\eta}$ into two indefinite parts. On the coset it is Lorentzian mostly minus, which means that $\tilde{g}_{00} = -1$ is fixed. To evaluate the sum $\mathcal{S}_n$ we now have to look more carefully. Although it is true that the summand again vanishes if and only if the two generators are the same, the product of the norms may not always be of the same sign now. It is thus the signs that we need to take care of. We do this by simply considering all possible combinations. Let us call generators which have negative Killing-square 'compact' and those with positive Killing-square 'non-compact'. Naturally this definition aligns with their (anti-)symmetry in the defining representations. Now, since we know how the Lorentz-algebra behaves, we know what type of generators will commute into each other. We have the following relations

$$[C, C] = C \qquad [-,-] = - \tag{5.46}$$
$$[C, \neg C] = \neg C \quad \Leftrightarrow \quad [-,+] = + \tag{5.47}$$
$$[\neg C, \neg C] = C \qquad [+,+] = - \tag{5.48}$$

where C and $\neg C$ refer to compact and non-compact respectively. Hence we obtain the following list. To

$\|[I,J]\|^2_{\tilde{\eta}}$	$\|I\|^2_{\tilde{\eta}}$	$\|J\|^2_{\tilde{\eta}}$	Π
$-$	$-$	$-$	$-$
$+$	$+$	$-$	$-$
$+$	$-$	$+$	$-$
$-$	$+$	$+$	$-$

our relief, we can infer that again the summand is identically -1. Thus the rest of the argument works out like in the hyperbolic case and we arrive at the potential

$$V_{dS_n}(\phi) = -\frac{1}{8}\mathcal{D}_n(\phi^2 - 1)^2 \tag{5.49}$$

which is the same as for the hyperbolic case.

Finally we have anti-de Sitter space AdS_n. Again the decomposition splits $\tilde{\eta}$ into two indefinite parts, with the coset part being Lorentzian. Though this time the latter is mostly plus, meaning that $\tilde{g}_{00} = +1$ is fixed. For the sum $\mathcal{S}_n$ we want to approach in the same way as for de Sitter space. This time though, it is not clear from the beginning what type of generators commute into each other. Thus, we first determine this by hand. The coset generators can be decomposed to

$$\mathfrak{m} = \text{span}(T) \oplus \text{span}\{X_b\}\,, \quad b = 1, ..., n-1 \tag{5.50}$$

with

$$T := \begin{pmatrix} 0 & 1 & \vec{0}^T \\ -1 & 0 & \vec{0}^T \\ \vec{0} & \vec{0} & 0 \end{pmatrix}\,, \quad (C) \tag{5.51}$$

$$X_b := \begin{pmatrix} 0 & 0 & \vec{e}_b^T \\ 0 & 0 & \vec{0}^T \\ \vec{e}_b & \vec{0} & 0 \end{pmatrix}\,, \quad (\neg C) \tag{5.52}$$

There are three types of commutators to consider, $[C, C]$, $[\neg C, C]$ and $[\neg C, \neg C]$. But we immediately see that the first case only consists of $[T, T] = 0$, since there is only one compact generator in $\mathfrak{m}$. Thus only the last two cases remain.

$$[T, X_b] = -\begin{pmatrix} 0 & 0 & \vec{0}^T \\ 0 & 0 & \vec{e}_b^T \\ \vec{0} & \vec{e}_b & 0 \end{pmatrix} = -\text{boost in } b \text{ direction} \tag{5.53}$$

The indices are now a bit awkward, since use greek ones for $\mathfrak{h}$, but we know what is meant. We conclude that $[C, \neg C] = \neg C$. Moving on we have

$$[X_a, X_b] = \begin{pmatrix} 0 & 0 & \vec{0}^T \\ 0 & 0 & \vec{0}^T \\ \vec{0} & \vec{0} & \vec{e}_{[a} \otimes \vec{e}_{b]}^T \end{pmatrix} \sim \text{rotation in } (a, b) \text{ plane} \tag{5.54}$$

where the indices are again awkward. In any case, we conclude $[\neg C, \neg C] = C$. Especially notice that we only get zero for trivial commutators. We can thus again write down a list of all possible combinations. Keeping in mind that compact with compact only yields zero because there only is one compact coset

$\|[I,J]\|^2_{\tilde{\eta}}$	$\|I\|^2_{\tilde{\eta}}$	$\|J\|^2_{\tilde{\eta}}$	Π
0	−	−	0
+	+	−	−
+	−	+	−
−	+	+	−

generator, we again see that the combinatorics works out the same way as for the two cases before. This time though $\tilde{g}_{00}$ is positive and thus we get for the potential

$$V_{AdS_n}(\phi) = +\frac{1}{8}\mathcal{D}_n(\phi^2 - 1)^2 \tag{5.55}$$

which this time is a usual double well potential.

With this we have successfully applied the CSDR scheme to the three Lorentzian cylinders over H^n, dS_n and AdS_n. Furthermore, all of our discussion also applies to the case $S^n \cong SO(n+1)/SO(n)$ discussed in [2] by dropping all terms coming from the conformal transformation which is being used.

Translating that case into our notation, the Lagrangian is of the exact same structure, with the same $\mathcal{D}_n$, while $\varepsilon_g = -1$, $\varepsilon_K = +1$ are implicitly being used and $\tilde{g}_{00} = +1$ which together renders the action negative. Since in that case $\tilde{g}_{00} = +1$ yielding a double well potential. We can thus give a closed expressions for all cases S^n, H^n ,dS_n and AdS_n at once, we have

$$\mathcal{L} = \frac{1}{\alpha}\varepsilon_K\,\tilde{g}_{00}\frac{n}{2}\mathcal{D}_n\left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right) \tag{5.56}$$

with

$$V(\phi) = \tilde{g}_{00}\frac{1}{8}\mathcal{D}_n(\phi^2-1)^2\,,\quad \mathcal{D}_n = 2(n-1) \tag{5.57}$$

and

$$\tilde{g}_{00} = \begin{cases} +1 \;\text{ for } S^n\,, AdS_n & \rightarrow \;\text{ double well} \\ -1 \;\text{ for } H^n\,, dS_n & \rightarrow \;\text{ inverted double well.} \end{cases} \tag{5.58}$$

Since the potentials only differ in their over all signs we arrive at the following picture. We can nicely

$$\begin{array}{ccc} V_{S^n} & = & -V_{H^n} \\ \shortparallel & & \shortparallel \\ V_{AdS_n} & = & -V_{dS_n} \end{array}$$

see the dualities of the spaces translating into the potentials due to the high symmetry of our setup.

5.3 Equations of motion

All systems (again including the case $S^n \cong SO(n+1)/SO(n)$) reduce to a single Newtonian degree of freedom ϕ subject to a double well- or inverted double well potential

$$V(\phi) = \pm\frac{1}{8}\mathcal{D}_n(\phi^2-1)^2 \tag{5.59}$$

which grows linearly in n with an over all factor of $\mathcal{D}_n = 2(n-1)$. The equations of motion thus are given by

$$\ddot{\phi} = -V'(\phi) = \pm\frac{1}{2}\mathcal{D}_n\phi(\phi^2-1) \tag{5.60}$$

which both can be solved analytically with the use of Jacobi elliptic functions. To do this we first have conservation of energy

$$\frac{1}{2}\dot{\phi}^2 + V(\phi) = E = \text{const.} \tag{5.61}$$

Hence we can fix a value for the energy E and without loss of generality choose $\dot{\phi}(0) = 0$ to solve the equations in terms of the energy and initial position.

We begin with the usual double well potential

$$\frac{1}{2}\dot{\phi}^2 + \frac{c}{2}(\phi^2-1)^2 = E \tag{5.62}$$

where we set $c := \frac{1}{4}\mathcal{D}_n$. Rescaling the foliation parameter as $u \mapsto t = \sqrt{c}\,u$ yields

$$\frac{1}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 + \frac{1}{2}(\phi^2-1)^2 = \frac{E}{c} =: \tilde{E}. \tag{5.63}$$

Where we introduced the scale adapted energy $\tilde{E} = E\,c^{-1} \geq 0$, where $E \geq 0$. Since the initial velocity is set to zero, we can define the initial position $\phi_* := \phi(0)$ via

$$\tilde{E} = \frac{1}{2}(\phi_*^2-1)^2. \tag{5.64}$$

This introduces ambiguity since we have to choose between the two possibilities

$$\phi_*^2 = 1 + \sqrt{2\tilde{E}} \quad \text{and} \quad \phi_*^2 = 1 - \sqrt{2\tilde{E}} \tag{5.65}$$

which correspond to the particle starting on the outer or inner side of the double well. Keeping this in mind we first proceed by not distinguishing the two. From the energy conservation we get

$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 = (\phi_*^2 - \phi^2)(\phi_*^2 + \phi^2 - 2). \tag{5.66}$$

Another rescaling with

$$y := \frac{\phi}{\phi_*} \ , \quad z := \frac{t}{\varepsilon} \tag{5.67}$$

then yields

$$\frac{1}{\varepsilon^2}\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2 = (1 - y^2)(\phi_*^2 + \phi^2 - 2). \tag{5.68}$$

We now want to tune ε in such a way that this becomes the defining equation for the Jacobi elliptic function $\mathrm{cn}(z, k^2)$, we get

$$\varepsilon^2(\phi_*^2 - \phi^2 - 2) = (1 + k^2 y^2 - k^2) \tag{5.69}$$

which is equivalent to

$$\varepsilon^2 \phi_*^2 = k^2 \ , \quad \varepsilon^2(\phi_*^2 - 2) = 1 - k^2. \tag{5.70}$$

Solving these equations is straight forward but we get two sets of solutions for the two initial conditions of ϕ_* (5.65) respectively.

$$\text{Outer}: \quad \varepsilon^2 = \frac{1}{2\sqrt{2\tilde{E}}} \ , \quad k^2 = \frac{1}{2} + \varepsilon^2 \tag{5.71}$$

$$\text{Inner}: \quad \bar{\varepsilon}^2 = \frac{-1}{2\sqrt{2\tilde{E}}} \ , \quad \bar{k}^2 = \frac{1}{2} + \bar{\varepsilon}^2 \tag{5.72}$$

Hence, switching outer to inner solution is equivalent to $\varepsilon^2 \mapsto -\varepsilon$ and $k^2 \mapsto 1 - k^2 = k'^2$, where the latter is just the dual modulus of k^2. Let us consider the behavior of the moduli depending on the scale adapted energy $\tilde{E}$. As we can see, we have for the outer solution $\tilde{E} \in (0, \frac{1}{2}) \leftrightarrow k^2 \in (\infty, 1)$ and $\tilde{E} \in (\frac{1}{2}, \infty) \leftrightarrow k^2 \in (1, \frac{1}{2})$, which for the former will replace the Jacobi elliptic function cd with dn. For the inner solution we have $\tilde{E} \in (0, \frac{1}{2}) \leftrightarrow k^2 \in (-\infty, 0)$ and $\tilde{E} \in (\frac{1}{2}, \infty) \leftrightarrow k^2 \in (0, \infty)$. For the latter range of energies there are no solutions beginning on the inner part of the double well and hence these ranges are excluded. Furthermore since $\bar{k}^2 = k'^2$ the region $\tilde{E} \in (0, \frac{1}{2})$ can be aligned for both inner and outer solutions. But since these solutions amount to the oscillations around one of the wells minima $\phi = \pm 1$, switching from outer to inner merely amounts to a half period shift, making the distinction of the two redundant in this case. Hence, sticking with the outer solutions only, we obtain all solutions for energies $\tilde{E} = \mathbb{R}_+ \setminus \{0, \frac{1}{2}\}$ by the Jacobi elliptic functions cn and dn. What is left are the limiting values for the energies, whose solutions can be easily guessed. We obtain for the full set of solutions of the double well

$$\phi(u) = \begin{cases} \frac{k}{\varepsilon}\mathrm{cn}\left(\frac{\sqrt{\mathcal{D}_n}}{2\varepsilon}\, u, k^2\right) & , \ \tilde{E} \in (\frac{1}{2}, \infty) \leftrightarrow k^2 \in (1, \frac{1}{2}) \\ \sqrt{2}\,\mathrm{sech}\left(\sqrt{\frac{\mathcal{D}_n}{2}}\, u\right) & , \ \tilde{E} = \frac{1}{2} \leftrightarrow k^2 = 1 \\ 0 & , \ \tilde{E} = \frac{1}{2} \leftrightarrow k^2 = 1 \\ \frac{k}{\varepsilon}\mathrm{dn}\left(\frac{\sqrt{\mathcal{D}_n}}{2\varepsilon}\, k\, u, \frac{1}{k^2}\right) & , \ \tilde{E} \in (0, \frac{1}{2}) \leftrightarrow k^2 \in (\infty, 1) \\ \pm 1 & , \ \tilde{E} = 0 \leftrightarrow k^2 = \infty \end{cases} . \tag{5.73}$$

The unstable solution $\phi \equiv 0$ corresponds to the Maurer-Cartan form of H $\mathcal{A} = I_\alpha e^\alpha$, which makes the field strength purely color-magnetic. While the two minima $\phi \equiv \pm 1$ correspond to two gauge equivalent versions of the Maurer-Cartan form of G, i.e. $\mathcal{A} = I_\alpha e^\alpha \pm I_a e^a$, making their respective field strengths vanish.

For the inverted double well

$$\frac{1}{2}\dot{\phi}^2 - \frac{c}{2}(\phi^2-1)^2 = E\ , \quad E \le 0 \tag{5.74}$$

we can absorb the negative sign of the potential via $c \mapsto -c$ making $\tilde{E} = Ec^{-1} \ge 0$ for $E \le 0$. Looking at the results for the non-inverted double well, we can see that this change ammounts to a Wick-rotation $u \mapsto i\,u$ which can be evaluated with identites of the Jacobi elliptic functions. This time though, the distinction between inner and outer starting solutions is not redundand anymore. As inner solutions will be bound and outer ones will run off to infinity, we have additional solutions to consider. Looking back at the non-inverted case, we have seen that $\phi_{in}(\bar{\varepsilon}, \bar{k}^2) = \phi_{out}(-\varepsilon, k'^2)$. Hence, the solutions for the inverted case are given by

$$\phi(u) = \begin{cases} \frac{k}{\varepsilon}\mathrm{nc}\left(\frac{\sqrt{\mathcal{D}_n}}{2\varepsilon}\,u, 1-k^2\right) & ,\ \tilde{E} \in (\frac{1}{2}, \infty) \leftrightarrow k^2 \in (1, \frac{1}{2}) \\ \sqrt{2}\sec\left(\sqrt{\frac{\mathcal{D}_n}{2}}\,u\right) & ,\ \tilde{E} = \frac{1}{2} \leftrightarrow k^2 = 1 \\ 0 & ,\ \tilde{E} = \frac{1}{2} \leftrightarrow k^2 = 1 \\ \frac{k}{\varepsilon}\mathrm{dc}\left(\frac{\sqrt{\mathcal{D}_n}}{2\varepsilon}\,k\,u, 1-\frac{1}{k^2}\right) & ,\ \tilde{E} \in (0, \frac{1}{2}) \leftrightarrow k^2 \in (\infty, 1) \\ i\frac{k'}{\varepsilon}\mathrm{dc}\left(\frac{i\sqrt{\mathcal{D}_n}}{2\varepsilon}\,k'\,u, \frac{1}{k^2}\right) & ,\ \tilde{E} \in (0, \frac{1}{2}) \leftrightarrow k^2 \in (\infty, 1) \\ \pm 1 & ,\ \tilde{E} = 0 \leftrightarrow k^2 = \infty \end{cases} \tag{5.75}$$

where

$$E = -\frac{\mathcal{D}_n}{4}\tilde{E} \le 0\ , \quad \varepsilon^2 = \frac{1}{2\sqrt{2\tilde{E}}}\ , \quad k^2 = \frac{1}{2} + \varepsilon^2\ , \quad k'^2 = 1 - k^2. \tag{5.76}$$

5.4 Energy momentum tensors

With the use of our closed expression (5.56), we can easily calculate the energy momentum tensors for all cases at once. We consider the standard Yang–Mills energy momentum tensor derived by varying the Lagrangian with respect to the inverse metric.

$$T_{\mu\nu} = \frac{1}{2\alpha}\left(\underbrace{K\left(F_{\mu\sigma}, F_{\nu\rho}\right)g^{\sigma\rho}}_{=:\tilde{T}_{\mu\nu}} - \frac{1}{4}g_{\mu\nu}K(F_{\alpha\beta}, F^{\alpha\beta})\right) \tag{5.77}$$

$$T = T_{\mu\nu}\, e^\mu \otimes e^\nu \tag{5.78}$$

For sake of convenience, we set $\alpha = \frac{1}{2}$ for the calculations. We start off by calculating the first part $\tilde{T}$.

$$\tilde{T}_{00} = K(F_{0\sigma}, F_{0\rho})g^{\sigma\rho} = K(F_{0a}, F_{0b})g^{ab} \tag{5.79}$$
$$= \dot{\phi}^2\mathcal{D}_n\varepsilon_K\tilde{\eta}_{ab}\varepsilon_g\tilde{\eta}^{ab} = n\mathcal{D}_n\varepsilon_k\varepsilon_g\dot{\phi}^2 \tag{5.80}$$

$$\tilde{T}_{ab} = K(F_{a\sigma}, F_{b\rho})g^{\sigma\rho} = \varepsilon_g\tilde{g}^{00}K(F_{a0}, F_{b0}) + K(F_{am}, F_{bn})g^{mn} \tag{5.81}$$
$$= \varepsilon_g\tilde{g}^{00}\varepsilon_K\mathcal{D}_n\tilde{\eta}_{ab}\dot{\phi}^2 + (\phi^2-1)^2\mathcal{D}_n\varepsilon_K\tilde{\eta}\left([I_a, I_m], [I_b, I_n]\right)\varepsilon_g\tilde{\eta}^{mn} \tag{5.82}$$

We again encounter a term which we can treat combinatorically.

$$\tilde{\eta}\left([I_a, I_m], [I_b, I_n]\right)\tilde{\eta}^{mn} = \sum_{I\in\mathfrak{m}}\tilde{\eta}\left([I, I_a], [I, I_b]\right)\ \|I\|^2_{\tilde{\eta}} \tag{5.83}$$
$$= \sum_{I\in\mathfrak{m}}\|[I, I_a]\|^2_{\tilde{\eta}}\,\|I\|^2_{\tilde{\eta}}\,\delta_{ab} =: \mathcal{C}_a\,\delta_{ab} \tag{5.84}$$

$\|[I_a]\|^2_{\tilde\eta}$	$\|I\|^2_{\tilde\eta}$	$\|[I,I_a]\|^2_{\tilde\eta}$	$\|I\|^2_{\tilde\eta}\,\|[I,I_a]\|^2_{\tilde\eta}$
$+$	$+$	$-$	$-$
$+$	$-$	$+$	$-$
$-$	$+$	$+$	$+$
$-$	$-$	$-$	$+$

Where in the last line we have used that the restriction $\mathrm{ad}(I) \equiv [I, \,] : \{I_a\} \to \{I_\alpha\}$ is injective. Again, we have to deal with the signs. From our computations before, we know that the relations (5.46)-(5.48) hold for all cases (the spheres included). We can thus again write down a list of combinations. We read off that for fixed a, the summand is either always -1 or $+1$, except once, for $I = I_a$ where it is zero. So we get

$$\mathcal{C}_a = (n-1)\begin{cases} -1\,, & a \text{ is } \neg C \\ +1\,, & a \text{ is } C \end{cases} \tag{5.85}$$

combining this with the δ_{ab} yields

$$\mathcal{C}_a \delta_{ab} = -(n-1)\tilde\eta_{ab} \tag{5.86}$$

Plugging this back in we arrive at

$$\tilde T_{ab} = \varepsilon_g \tilde g^{00} \varepsilon_K \mathcal{D}_n \tilde\eta_{ab}\dot\phi^2 - (\phi^2-1)^2 \mathcal{D}_n \varepsilon_K \varepsilon_g (n-1)\tilde\eta_{ab} \tag{5.87}$$

$$= \varepsilon_K \mathcal{D}_n g_{ab} \tilde g_{00}\left(\dot\phi^2 - 4V(\phi)\right) \tag{5.88}$$

where in the last step we have used $\tilde g_{00}^2 = 1$. And finally the other mixed components

$$\tilde T_{0a} \sim K(\mathfrak{h}, \mathfrak{m}) \equiv 0 \tag{5.89}$$

all vanish. Moving on, the second part of the energy momentum tensor is proportional to the lagrangian.

$$-\frac{1}{4} g_{\mu\nu} K(F_{\alpha\beta}, F^{ab}) = -2 g_{\mu\nu}\mathcal{L} \tag{5.90}$$

$$= -g_{\mu\nu}\varepsilon_K n \mathcal{D}_n \tilde g_{00}\left(\frac{1}{2}\dot\phi^2 - V(\phi)\right) \tag{5.91}$$

Together with the other part $\tilde T$, we thus obtain

$$T_{00} = n\mathcal{D}_n \varepsilon_K \varepsilon_g (\tilde g_{00})^2 \dot\phi^2 - \varepsilon_K \tilde g_{00} g_{00} n \mathcal{D}_n \left(\frac{1}{2}\dot\phi^2 - V(\phi)\right) \tag{5.92}$$

$$= \varepsilon_K \tilde g_{00}\, n\mathcal{D}_n \left(\frac{1}{2}\dot\phi^2 + V(\phi)\right) g_{00} =: \varepsilon_K \tilde g_{00}\, n\mathcal{D}_n E g_{00} \tag{5.93}$$

and

$$T_{ab} = \varepsilon_K \mathcal{D}_n g_{ab}\tilde g_{00}\left(\dot\phi^2 - 4V(\phi)\right) - \varepsilon_K n \mathcal{D}_n \tilde g_{00} g_{ab}\left(\frac{1}{2}\dot\phi^2 - V(\phi)\right) \tag{5.94}$$

$$= -\varepsilon_K \tilde g_{00}\mathcal{D}_n \left(\left(\frac{n}{2}-1\right)\dot\phi^2 + (4-n)V(\phi)\right) g_{ab}. \tag{5.95}$$

And again with the same argument before we conclude that $T_{0a} \equiv 0$. Hence, we have derived the energy momentum tensors for all cases. They all almost look the same, the only difference being the sign given by $\tilde g_{00}$. Written down neatly and reintroducing the coupling α, we have

$$T_{00} = \frac{1}{2\alpha}\varepsilon_K\, n\mathcal{D}_n \tilde g_{00} E\, g_{00} \tag{5.96}$$

$$T_{ab} = -\frac{1}{2\alpha}\varepsilon_K \tilde g_{00}\mathcal{D}_n \left(\left(\frac{n}{2}-1\right)\dot\phi^2 + (4-n)V(\phi)\right) g_{ab} \tag{5.97}$$

$$T_{0a} = 0. \tag{5.98}$$

Notice that the sign of T depends both on the overall sign of the metric (which is expected) and also on the sign of the Killing form ε_K. Hence, there is freedom to choose (especially) the sign of the energy density. A practical choice may be to set $\varepsilon_K = -\tilde{g}_{00}$. This choice is already suggested by the case of the spheres. In that case $\tilde{g}_{00} = +1$ is positive since the structure group $SO(n+1)$, and hence the coset, is compact. In that case choosing the negative of the Killing form, i.e. $\varepsilon_K = -1 = -\tilde{g}_{00}$ always results in non-negative action, as it is usually chosen. The suggested choice indeed will also always keep the action positive for the other cases which can be seen by our general expression for the Lagrangians (5.56), when taking into account the respective orientations of the double wells. We will later on see another reason why this choice may be more reasonable.

For the particular cases of Riemannian cosets S^n and H^n the the spacetimes are homogeneous and isotropic. In this case the Yang–Mills fields actually yield perfect fluid energy momentum tensors. Of course, this is not true anymore for the slicings with the Lorentzian cosets $(A)dS_n$. Especially note on the latter that, since the foliation parameter is spacelike in those cases, the energy density will not reside in T_{00} but in T_{aa} for the timelike a. For $n = 3$, that is, spacetime dimension four, the energy momentum tensors read

$$T_{00} = \frac{1}{2\alpha}\varepsilon_K 12\tilde{g}_{00}E\, g_{00} \tag{5.99}$$

$$T_{ab} = -\frac{1}{2\alpha}\varepsilon_K 3\tilde{g}_{00}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) g_{ab} \tag{5.100}$$

$$= -\frac{1}{2\alpha}\varepsilon_K \tilde{g}_{00} 3\, g_{ab} \tag{5.101}$$

Then with $\rho := \frac{1}{2\alpha}\varepsilon_K n\mathcal{D}_n\tilde{g}_{00}E$ this becomes

$$T = \rho\left(g_{00}e^0 \otimes e^0 - \frac{1}{3}g_{ab}e^a \otimes e^b\right) \tag{5.102}$$

again matching [5] for hyperbolic- and de Sitter space. The energy momentum tensors are traceless (as expected) and for the Riemannian slicings with S^3 and H^3 also of perfect fluid radiation type.

6 Putting the solutions to use

When looking at our CSDR setup, it becomes evident that there is the possibility to generalize and couple to gravity rather easily as it would be the case when dealing with non-symmetric scenarios. In particular two of our four spacetimes are homogeneous and isotropic which suggests a coupling to FLRW type geometries, like e.g. [10], [8]. Furthermore, many manifolds of the 'not too exotic' type can be obtained by our cylinders by either introducing a warping, gluing different parts of the cylinders together or a combination of these both like it was done for example in [5] or [2]. Hence it is of interest to investigate these modifications to our setup, which is the topic of this chapter. We will couple our system to open type hyperbolic cosmology in four dimensional GR in the first part. In the second part we will derive the equations of motion for a general warping function and in the last part we will apply those results to the hyperbolic foliation of anti-de Sitter space.

6.1 Hyperbolic FLRW cosmology

At the end of the last chapter, we saw that for the two cases S^3 and H^3 the energy momentum tensor has the same structure as for perfect fluid radiation in cosmology. Indeed, the spacetime in these cases are conformally equivalent to two of the three standard FLRW models used in cosmology. FLRW type closed for S^3 and FLRW type open (and hyperbolic) for H^3. Together with the spacelike slicing and symmetry constraint used in the CSDR construction the Yang–Mills fields become not only invariant under action of the big group but in particular also under the little sub group H, which is $SO(3)$ for both S^3 and H^3. This means that the solutions are by construction adapted to the cosmological principle, which is reflected in the perfect fluid structure of their energy momentum tensors. So at first glance, one might try to couple the Yang–Mills system to the Einstein equations with the corresponding FLRW ansatz. Indeed, such a coupling is possible. Even more so, since FLRW only has one degree of freedom, namely the scale factor $a(t)$, which only appears as a conformal factor in the metric, the Yang–Mills system does not feel the dynamics of $a(t)$. That is, the coupling of the two systems is a one way coupling.

$$\text{Yang–Mills} \;\underset{\not\; a(t)}{\overset{T_{\mu\nu}}{\rightleftarrows}}\; \text{Einstein}$$

This provides the opportunity to - in principle - get analytic solutions to the coupled Einstein–Yang–Mills System. The procedure is straight forward; Solve YM, calculate $T_{\mu\nu}$, plug it into the Einstein equations and solve the Einstein equations. Exactly this has been done for the $S^3 \cong SU(2)$ case in [10], [8] and we now will do the same thing for the hyperbolic CSDR case.

The standard, co-moving time representation of FLRW reads

$$g = -\,\mathrm{d}t \otimes \mathrm{d}t + a^2(t)\, g_{\tilde{M}} \tag{6.1}$$

where $g_{\tilde{M}}$ is the 'canonical' $SO(3)$ invariant metric of constant normalized sectional curvature for the three cosmologies $\tilde{M} = S^3, H^3, \mathbb{R}^3 \doteq \sec = +1, -1, 0$. That is, $g_{\tilde{M}}$ is just (up to sign) $\tilde{\eta}_{ab} e^a \otimes e^b$ from before. In these coordinates the Einstein tensor becomes

$$G^0_{\ 0} = -3\{\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\} \tag{6.2}$$

$$G^a_{\ b} = -\{2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\}\,\delta^a_{\ b} \tag{6.3}$$

where the dot denotes $\partial_0 \equiv \partial_t$ and $k = \pm 1, 0$ is the (normalized) sectional curvature. Now, since FLRW is just a warped cylinder over $\tilde{M}$, we can always make it conformally flat by introducing conformal time τ.[13]

$$\mathrm{d}\tau = \frac{1}{a(t)}\mathrm{d}t \quad \Leftrightarrow \quad \tau(t) = \int \mathrm{d}t' \frac{1}{a(t')} \tag{6.4}$$

Hence, the metric now reads

$$g = a^2(\tau)\,(-\,\mathrm{d}\tau \otimes \mathrm{d}\tau + g_{\tilde{M}}) \tag{6.5}$$

[13] sometimes also called redshift time or arc parameter

The derivatives of the scale factor transform to

$$\partial_\tau = a\partial_t \ \Rightarrow \dot{a} = \frac{a'}{a} \ , \ \ \ddot{a} = \frac{a''}{a^2} - \frac{(a')^2}{a^3} \tag{6.6}$$

where the prime now denotes differentiation with respect to conformal time ∂_τ.

The Einstein equations or in this context also called Friedmann equations

$$G + \Lambda g = \kappa T \ , \ \ G = Ric - \frac{1}{2}\mathcal{R}g \ , \ \ \mathcal{R} = \mathrm{tr}_g(Ric) \tag{6.7}$$

in the vicinity of a traceless perfect fluid are equivalent to the trace part and (00)-part of the equation.

$$\mathrm{tr}_g(G + \Lambda g) = 0 = -\mathcal{R} + 4\Lambda \tag{6.8}$$
$$G_{00} + \Lambda g_{00} = \kappa T_{00} \tag{6.9}$$

Expressing these in conformal time is straight forward. Starting with the trace, we have

$$\mathrm{tr}_g(G + \Lambda g) = G^0_{\ 0} + 3G^1_{\ 1} + 4\Lambda \tag{6.10}$$
$$= -3\{\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\} - 3\{2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\} + 4\Lambda \tag{6.11}$$
$$= -\{3\left(\frac{a'}{a^2}\right)^2 + 3\frac{k}{a^2} + 6\frac{a''}{a^3} - 3\left(\frac{a'}{a^2}\right)^2 + 3\frac{k}{a^2}\} + 4\Lambda \tag{6.12}$$
$$= \underbrace{-6\{\frac{a''}{a^3} + \frac{k}{a^2}\}}_{-\mathcal{R}} + 4\Lambda = 0 \tag{6.13}$$

which is equivalent to

$$a'' = -ka + \frac{4}{6}\Lambda a^3 = \partial_a\left(-\frac{k}{2}a^2 + \frac{\Lambda}{6}a^4\right) =: -W'(a) \ , \ \ W(a) := \frac{k}{2}a^2 - \frac{\Lambda}{6}a^4. \tag{6.14}$$

For the (00)-part we need to consider the energy momentum tensor. We identify the conformal time τ with the foliation parameter u of the Yang–Mills setup which transforms the energy-momentum tensor to

$$T = -\frac{\rho}{a^2}(\,\mathrm{d}\tau \otimes \mathrm{d}\tau + \frac{1}{3}g_{\tilde{M}}). \tag{6.15}$$

With $\rho = \frac{1}{2\alpha}\varepsilon_K 3\mathcal{D}_3\tilde{g}_{00}E$ and the overall minus coming from the fact that we are now explicitly working in mostly plus signature. Pulling down the index of the Einstein tensor and plugging in the energy momentum tensor then yields

$$G_{00} + \Lambda g_{00} = \kappa T_{00} \tag{6.16}$$
$$\Leftrightarrow \{3\dot{a}^2 + k\} - \Lambda a^2 = -\frac{\kappa\rho}{a^2} \tag{6.17}$$
$$\Leftrightarrow 3\left(\frac{a'}{a}\right)^2 + 3k - \Lambda a^2 = -\frac{\kappa\rho}{a^2} \tag{6.18}$$
$$\Leftrightarrow \frac{1}{2}(a')^2 + \frac{k}{2}a^2 - \frac{\Lambda}{6}a^4 = -\frac{1}{6}\kappa\rho \tag{6.19}$$
$$\Leftrightarrow \frac{1}{2}(a')^2 + W(a) = -\frac{1}{6}\kappa\rho =: E_{\mathrm{GR}}. \tag{6.20}$$

To avoid confusion, we now denote differentiation with respect to *conformal time* τ with an over dot. Thus the equations of motion for the scale factor read

$$\ddot{a} + W'(a) = 0 \tag{6.21}$$

$$\frac{1}{2}\dot{a}^2 + W(a) = -\frac{\kappa}{\alpha}\varepsilon_K\tilde{g}_{00}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \ \ \Leftrightarrow \ \ E_{\mathrm{GR}} = -\frac{\kappa}{\alpha}\varepsilon_K\tilde{g}_{00}E_{\mathrm{YM}}. \tag{6.22}$$

Hence, the dynamics of the scale factor is, like for the Yang–Mills system, that of a Newton like particle subject to a cosmological potential $W(a)$. The coupling to Yang–Mills is solely through the 'energy balancing' (6.22) which is also known as the Wheeler De-Witt constraint. The cosmological potential $W(a)$, and hence the range of possible dynamics of the spacetime, is determined by the sectional curvature k of the spatial slicing and the sign of the cosmological potential.

$$W(a) = \frac{k}{2}a^2 - \frac{\Lambda}{6}a^4 \tag{6.23}$$

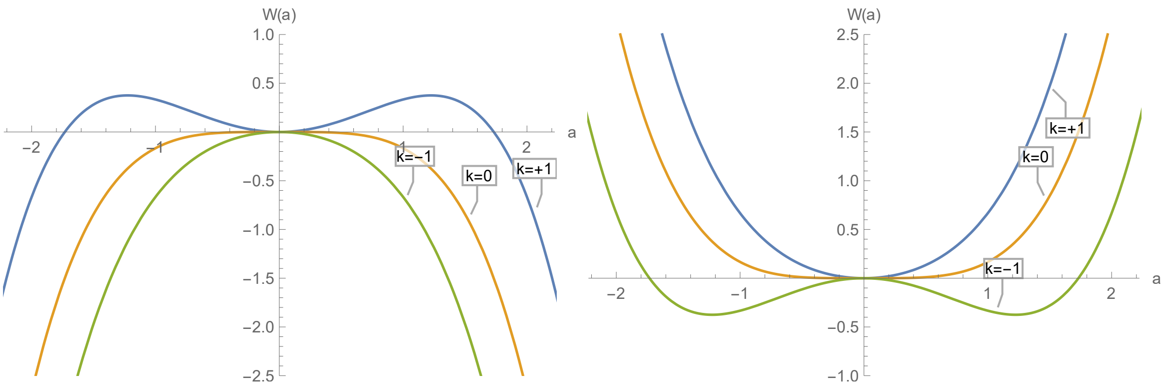

Figure 1: Plots of the cosmological potential $W(a)$ for different values of $k = \pm 1, 0$ and cosmological constant $\Lambda = +1$ (left), $\Lambda = -1$ (right).

In [10] ,[8] the case of $k = +1 \leftrightarrow \tilde{M} = S^3$ with $\Lambda > 0$ was considered. In that case, the Yang–Mills system was subject to a double well and the scale factor to an inverted double well, resulting either in periodic or in blow-up solutions for $a(\tau)$. Now, for the case $k = -1 \leftrightarrow \tilde{M} = H^3$, that is, open type hyberbolic cosmolgies, the Yang–Mills system is subject to an inverted double well and the dynamics of the scale factor depends on the sign of Λ. For the latter there are two possibilities:

(i) $\Lambda > 0 \;\Rightarrow\; W(a)$ strictly monotonously decreasing $\rightarrow$ Only unstable, blow-up solutions.

(ii) $\Lambda < 0 \;\Rightarrow\; W(a)$ double well $\rightarrow$ Oscillatory solutions both with and without 'big crunch' and stationary solution at potential minima.

Focusing on the case $\Lambda < 0$; By again fixing $\dot{a}(0) = 0$ the solutions for the scale factor in conformal time can be parametrized by the initial energy E_{GR} and initial position $a(0)$.

$$a(\tau) = \begin{cases} \sqrt{\frac{-3}{2\Lambda}\frac{k}{\varepsilon}}\,\text{cn}\left(\frac{1}{\sqrt{2\varepsilon}}\,\tau, k^2\right) & , \; E_{GR} \in (\infty, 0) \leftrightarrow k^2 \in (\frac{1}{2}, 1) \\ \sqrt{\frac{-3}{\Lambda}}\text{sech}\left(\frac{\tau}{2}\right) & , \; E_{GR} = 0 \leftrightarrow k^2 = 1 \\ 0 & , \; E_{GR} = 0 \leftrightarrow k^2 = 1 \\ \sqrt{\frac{-3}{2\Lambda}\frac{k}{\varepsilon}}\,\text{dn}\left(\frac{1}{\sqrt{2\varepsilon}}k\,\tau, \frac{1}{k^2}\right) & , \; E_{GR} \in \left(0, \frac{3}{8\Lambda}\right) \leftrightarrow k^2 \in (1, \infty) \\ \pm\sqrt{\frac{-3\Lambda}{2}} & , \; E_{GR} = \frac{3}{8\Lambda} \leftrightarrow k^2 = \infty \end{cases} \tag{6.24}$$

With

$$E_{GR} \in \left(\frac{3}{8\Lambda}, \infty\right) \;,\; \varepsilon^2 = \frac{1}{2\sqrt{1 - \frac{8\Lambda}{3}E_{GR}}} \;,\; k^2 = \frac{1}{2} + \varepsilon^2. \tag{6.25}$$

Going back to the energy balancing (6.22), we notice that there is a freedom to choose the sign ε_K with which the Yang–Mills energy couples to the spacetime dynamics. As mentioned before, a natural choice may be $\varepsilon_K = -\tilde{g}_{00}$ which preserves the sign of E_{YM}. Indeed, for the bounded solutions $\phi(\tau)$ of the inverted double well the energy is always non-positive thus yielding bounded solutions around the local minima for the scale factor. More precisely we have that

$$E_{\text{YM}} \in [-\frac{1}{2}, 0] \quad \Leftrightarrow \quad -\frac{\alpha}{\kappa}\varepsilon_K \tilde{g}_{00}\, E_{\text{GR}} \in [-\frac{1}{2}, 0] \tag{6.26}$$

which yields two possibilities:

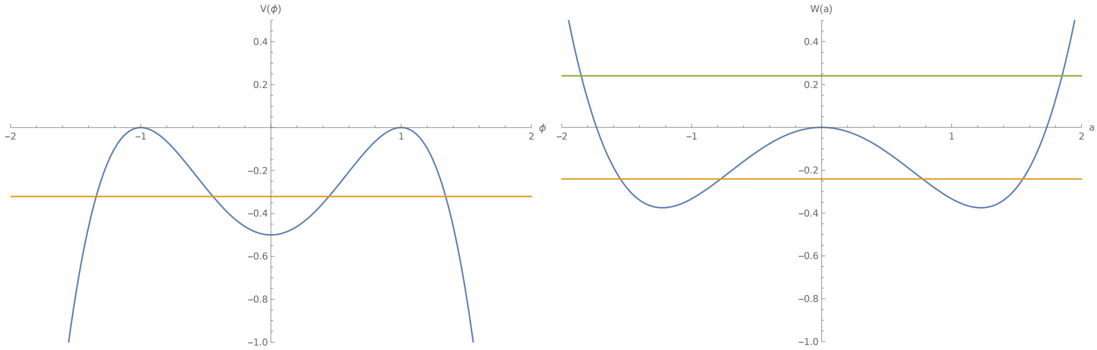

Figure 2: Energy balancing of YM field and scale factor depending on the choice of $\varepsilon_K = \pm\tilde{g}_{00}$ for $\Lambda = -1$, $\alpha = \frac{8}{6}$ (minimal) and $\kappa = 1$.

(i) $\varepsilon_K = -\tilde{g}_{00} \rightarrow$ energy couples directly $E_{\mathrm{GR}} \in [-\frac{\kappa}{2\alpha}, 0]$

(ii) $\varepsilon_K = +\tilde{g}_{00} \rightarrow$ energy couples inverted $E_{\mathrm{GR}} \in [0, \frac{\kappa}{2\alpha}]$

The second choice is, of course, always possible. If big bang initial conditions are chosen, the solution becomes trivial $a \equiv 0$, whereas starting with non-zero scale factor yields one solution at $E_{\mathrm{GR}} = 0$ with shrinking scale factor reaching zero at infinite conformal time and a set of solutions oscillating through the well passing through zero (big crunch). The first choice on the other hand provides a relation between the cosmological constant Λ and the Yang–Mills coupling α. Since the energy of the scale factor is bounded from below by $E_{\mathrm{GR}}^{(\mathrm{crit})} = \frac{3}{8\Lambda} < 0$, we may not allow arbitrarily low values for the Yang–Mills energy. If we still want to allow for the Yang–Mills field to be able to sit in its local minimum, we get the condition

$$\frac{3}{8\Lambda} \leq -\frac{\kappa}{2\alpha} \quad \Leftrightarrow \quad \alpha \geq -\frac{1}{2}\frac{8\Lambda}{3\kappa}. \tag{6.27}$$

If the condition is guaranteed (either by choosing Λ or α appropriately), any stable Yang–Mills solution will yield a sensible cosmological solution. The latter being oscillations around one of the minima of $W(a)$.

6.2 The case of general warping

Up until now we have only considered trivial products $\mathbb{R} \times G/H$ for our cylinders, that is, the metric did not include a warping function. The simplicity of our particular CSDR construction makes it easy to generalize to the warped case which is the topic of this subsection.

The metric of a generally warped cylinder $\mathbb{R} \times_a \tilde{M}$ with warping function $a(u)$ is given by

$$g = \mathrm{d}u \otimes \mathrm{d}u + a^2(u) g_{\tilde{M}}. \tag{6.28}$$

As we have seen before, we can always make this conformally flat by introducing conformal 'time' τ (the signature of the metric does not play a role here) via $\mathrm{d}\tau = \frac{1}{a}\mathrm{d}u$.

$$g = a^2(\tau)(\mathrm{d}\tau \otimes \mathrm{d}\tau + g_{\tilde{M}}) \tag{6.29}$$

Since this is always possible, we will restrict ourselves just to conformal transformations. Now let g be the flat cylinder metric of our CSDR construction (5.31). Under a conformal transformation of the form

$$g \mapsto e^{2\sigma(u)} g \tag{6.30}$$

the reduced action (5.35) transforms to

$$S[\phi] = \mathrm{Vol}(G/H) \int_{\mathbb{R}} \mathcal{L}\mathrm{d}u \mapsto \mathrm{Vol}(G/H) \int_{\mathbb{R}} e^{(n-3)\sigma(u)} \mathcal{L}\mathrm{d}u =: S^{(\sigma)}[\phi]. \tag{6.31}$$

By introducing the 'conformal Hubble parameter' $\mathcal{H}(u)$ as

$$\mathcal{H}(u) := e^{-\sigma(u)} \frac{\mathrm{d}}{\mathrm{d}u} e^{\sigma(u)} = \dot{\sigma}(u) \tag{6.32}$$

we can express the equations of motion for the warped case as

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\mathrm{d}}{\mathrm{d}u} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = (n-3)\mathcal{H}(u)\dot{\phi} \tag{6.33}$$

which is nothing but

$$\ddot{\phi} + V'(\phi) = (n-3)\mathcal{H}(u)\dot{\phi}. \tag{6.34}$$

We see that the conformal invariance of Yang–Mills in four spacetime dimensions is nicely captured in the factor $(n-3)$. Thus, the introduction of a warping function - or scale factor respectively - changes the equations of motion by an additional Hubble friction term.

Of course, there is nothing much to say about this equation for an arbitrary Hubble parameter. Even if we linearize the equations by only considering small pertubations around the minima of the quartic potentials, the 'time' dependence of the friction term makes the system still not easy to solve. A special case would be to consider constant Hubble parameter which is equivalent to $\sigma = \lambda u$ for some $\lambda \in \mathbb{R}$, that is, exponentially growing or shrinking universe. In this case the pertubations amount to a simple harmonic oscillator with constant (albeit possibly negative) friction, which is trivially solvable.

6.3 Warping to AdS_n

In [2], i.e. the case S^n, the product $\mathbb{R} \times G/H$ is not trivial but warped. The reason being that with the warping function $\cosh^2 \tau$, the product $\mathbb{R} \times S^n$ becomes de Sitter space dS_{n+1}. That is

$$g_{dS_{n+1}} = (-\,\mathrm{d}\tau \otimes \mathrm{d}\tau + \cosh^2 \tau g_{S^n}) \tag{6.35}$$

Even more so, this is conformally equivalent to the flat (finite) cylinder via

$$t = \arctan(\sinh \tau)\ , \quad t \in \mathcal{I} := (-\frac{\pi}{2}, \frac{\pi}{2}) \tag{6.36}$$

$$g_{dS_{n+1}} = \frac{1}{\cos^2 t}(-\,\mathrm{d}t \otimes \mathrm{d}t + g_{S^n}) \tag{6.37}$$

Of course, the conformal factor drops out of the action in four spacetime dimensions, i.e. for $n = 3$. For $n \neq 3$, the warping does play a role and at the end amounts to a friction term for $\phi(t)$

$$\ddot{\phi} = -(n-3)\tan t\, \dot{\phi} - V'_{\mathbb{S}^n}(\phi) \tag{6.38}$$

which aligns with our general formula (6.34). Considering the dynamics of this system for $t \geq 0$ we have that for $n < 3$ the friction is not dissipative and and diverges at finite time, hence all solutions except the constant ones blow up. For $n > 3$ the friction becomes dissipative, i.e. the 'particle' loses energy along its evolution. In this case, since the double well of the spherical case is non-inverted, all solutions get dampened to a total standstill.

Similarly how H^n and S^n are dual to each other, the same goes for AdS_n and dS_n. Likewise, in the same way as dS_{n+1} can be obtained by a spacelike S^n slicing, AdS_{n+1} can be obtained by a spacelike H^n slicing. In the hyperbolic slicing coordinates, the metric of AdS_{n+1} reads

$$g_{AdS_{n+1}} = (-\,\mathrm{d}\tau \otimes \mathrm{d}\tau + \cos^2 \tau g_{H^n}) \tag{6.39}$$

and as before we can introduce conformal time

$$t = \operatorname{arctanh}(\sin \tau)\ , \quad t \in \mathbb{R} \tag{6.40}$$

making it conformally equivalent to the flat cylinder

$$g_{AdS_{n+1}} = \frac{1}{\cosh^2 t}(-\,\mathrm{d}t \otimes \mathrm{d}t + g_{H^n}). \tag{6.41}$$

Hence, in the same way as for the spheres, we obtain a variety of equations on AdS_n for free by simply warping the H^n case. Using our general result again we can read off the Hubble friction and thus get

$$\ddot{\phi} = (n-3)\tanh t\,\dot{\phi} - V'_{H^n}(\phi). \tag{6.42}$$

Naturally, this equation looks similar to the one obtained by the spherical slicing of dS_{n+1}. There are three main things different to the previous case. Firstly, the friction coefficient stays bounded, secondly, dissipative is now for $n < 3$ and non-dissipative for $n > 3$ and thirdly, the potential is an inverted double well. Focusing again on $t \geq 0$, since the potential is inverted, *practically* no initial condition for $n > 3$ will stay bounded. For $n < 3$, in particular $n = 2$ (since the potential vanishes for $n = 1$), we have a 'particle' subject to an inverted double well with growing but bounded dissiptaive friction. Due to the invertedness of the potential, not all initial conditions will result in the particle being dampened to still stand. Only for a portion $\Omega \subset \{(\phi_0, \dot{\phi}_0)\}$ inside the phase space of initial conditions this will happen. One can get an idea of this portion naturally by considering the situation where the particle starts outside the well but is shot into it with such a high (but not too high) velocity that by the time it reaches the inside of the well its energy has been dissipated enough for it to stay bounded. We can numerically evaluate this region of initial conditions Ω for which the particle stays bounded for different initial times $t_0 \geq 0$. The algorithm numerically evolves the initial condition and stops the simulation

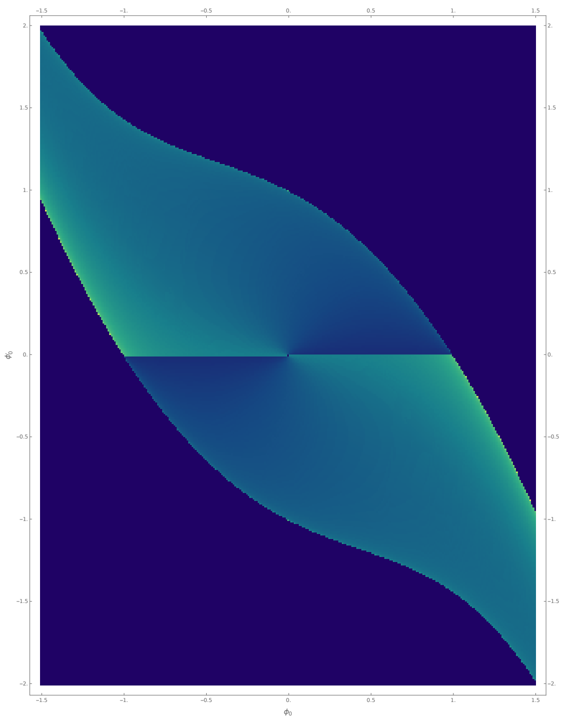

Figure 3: Numerically sampled region of initial conditions for bounded solutions starting at $t_0 = 0$.

when the direction of velocity changes *and* the position is inside the well. The time at which this event is triggered is then captured and plotted. Hence, the shading of the plot shows how fast the particle reaches the triggering event. There are more convenient measures for when the solution will stay bounded, e.g. when the energy is less then zero *and* the position inside the well, though this approach sometimes has problems, for example when them initial conditions already satisfy the triggering condition.

We can also look at different initial times, which will change the region due to the time dependence of the friction coefficient.

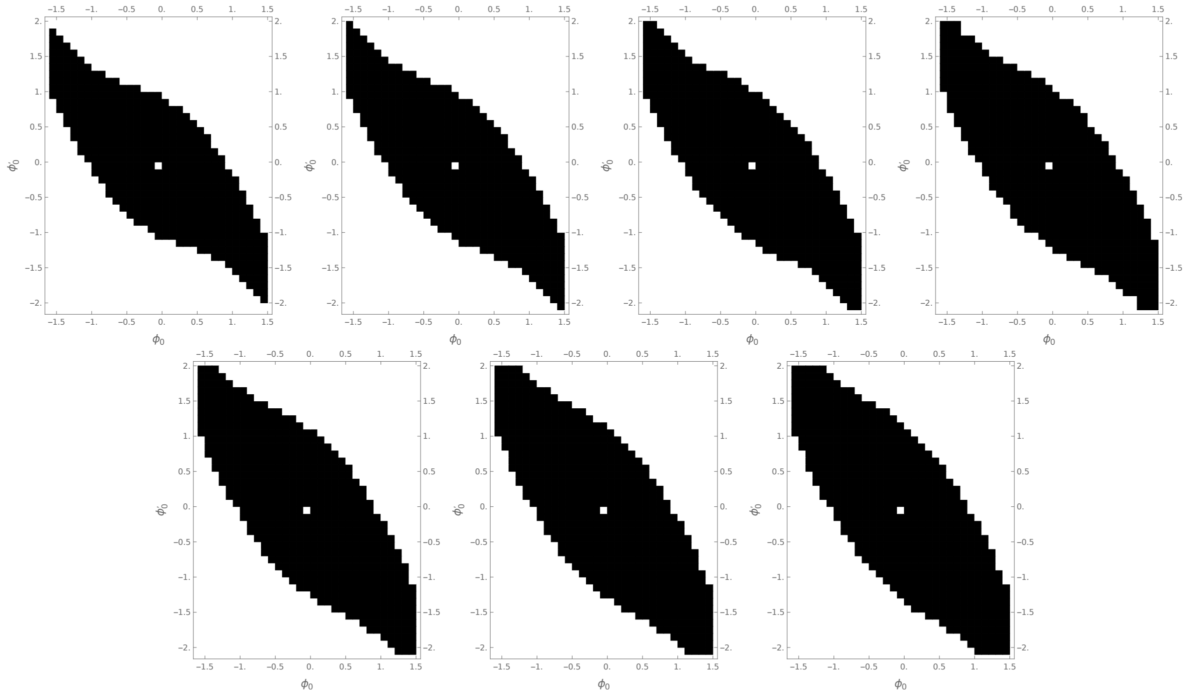

Figure 4: Numerically sampled regions of initial conditions for bounded solutions with lower resolution for starting times $t_0 \in [0, 0.5]$ with step size of $dt = 0.1$. The dot in the origin is just an artifact of the code, since solutions starting at rest in an extremum wont move at all.

7 Conclusion and outlook

We have implemented the coset space dimensional reduction scheme for cylinders over the three non-compact symmetric spaces hyperbolic space H^n, de Sitter space dS_n and anti-de Sitter space AdS_n. We found that all three cases lead to the same reduced action of a Newton-like particle subject to a quartic potential, with the only difference being the orientation of said potential depending on the coset. In this we could also include the previously done case of the spheres S^n [2]. We also obtained a closed expression for the energy-momentum tensors of all cases at once. Together with the latter we were also able to couple the four dimensional hyperbolic case to gravity in an analogous way to as it was done for the $S^3 \cong SU(2)$ case [10], [8]. Furthermore we generalized the results to the case of arbitrary warping of the cylinders which resulted in a Hubble friction term for the equations of motion. Utilizing the latter we were able to obtain the equations of motion for solutions on AdS_n coming from the hyperbolic slicing coordinates, which is the dual case to [2]. Since the equations of motion for the unwarped cases were just those of a particle subject to a (inverted-) double well potential, we managed to derive an infinite family of analytic Yang–Mills solutions on cylinders over H^n, dS_n and AdS_n of arbitrary dimension.

At this stage there are further generalizations and properties that could be of interest for future works. First of all considering only the solutions already obtained, one could investigate the power spectra or stability of the solutions. The latter was considered for $S^3 \cong SU(2)$ case in [8], [24]. Then there is the possibility to write the symmetric spaces at hand as non-symmetric cosets which will in general lead to more than one degree of freedom and hence richer dynamics. In the same vein, other symmetric spaces could also be investigated. Considering the generalization to warped cylinders, there is the possibility to in principle instantly write down the equations of motion for any particular spacetime which is conformal to our cylinders. Especially the coupling to FLRW in higher dimensional gravity may be of interest as the equations look familiar to ones obtained when studying inflation.

A Appendix

A.1 Mathematica code to evaluate the region of bounded initial conditions

```
In[1]:= ClearAll[DoArrayPlot];
DoArrayPlot[{t0_, T_}, n_, {dx_, dv_}] :=
 Module[{startpos, X, startvel, V, inttime, inttime0, sols},
  startpos = Table[x, {x, -1.5, 1.5, dx}];
   X = Length[startpos];
   startvel = Table[v, {v, 2, -2, -dv}];
   V = Length[startvel];

   inttime0 = Table[Table[t0, {v, 1, V}], {x, 1, X}];
   inttime = inttime0;

   sols = Table[Table[

      NDSolve[{y''[t] == (n - 1) y[t] (y[t]^2 - 1) + (n - 3) Tanh[t] y'[t] ,
        y[t0] == startpos[[x]],  y'[t0] == startvel[[v]],
        WhenEvent[Sign[y'[t]] == -Sign[startvel[[v]]] && Abs[y[t]] < 1 ,
         inttime[[x, v]] = t ;  "StopIntegration"]},
       y , {t, t0, T}]

      , {v, 1, V}], {x, 1, X}]; // Quiet;

  tempValues = Abs@Transpose[inttime - inttime0];
  tempValues = Table[If[tempValues[[i, j]] == 0, 0, tempValues[[i, j]] + 1.5], {i, V}, {j, X}];
  tempBooleans = Table[Boole[tempValues[[i, j]] > 0], {i, 1, V}, {j, 1, X}];

  tempGraph = ArrayPlot[tempValues, ColorFunction → "BlueGreenYellow"];
  Show[tempGraph, FrameLabel -> {Style["ϕ0", 14], Style["ϕ0'", 14]},
   Frame -> True, FrameTicks -> {Table[{k, startpos[[k]]}, {k, 1, X, Floor[X / 6]}],
     Table[{k, -startvel[[k]]}, {k, 1, V, Floor[V / 8]}]}]
 ]
```

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