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Search for a Casimir Operator in Calogero Models

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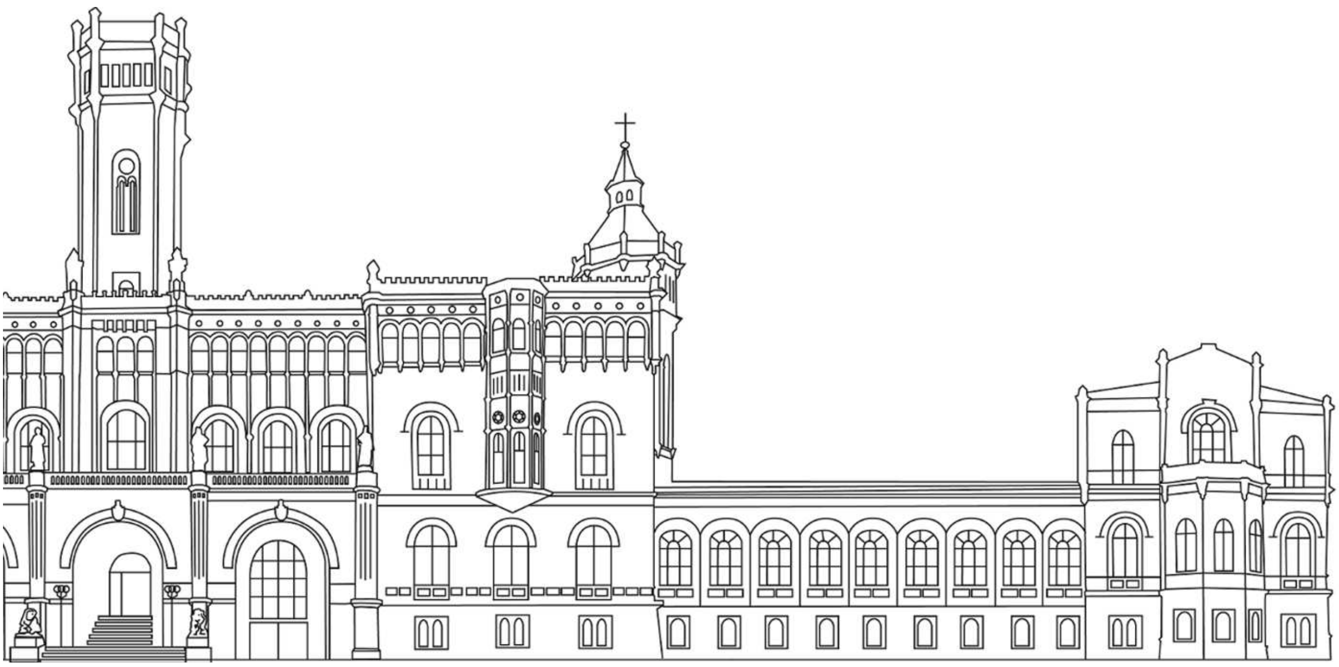

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Abstract

A Casimir operator for the Calogero model with three particles and interaction proportional to the inverse of the square of the distance between two particles was determined, using B_{kl}-operators constructed from the position and momentum operators, which are symmetric under particle permutations, and satisfy a $W_{1+\infty}$ algebra. Applying simple algebraic transformations and properties of Weyl ordering, the center-of-mass and total momentum operators were successfully decoupled from the Casimir operator. The final expression includes the effects of the interaction, and contains the classical limit $\hbar \to 0$; detailed calculations and proofs can be found in the appendix.

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1 Introduction

Many-particle problems appear naturally in Physics, from molecular interactions up to cosmological processes. As every student of physics knows, many-particle problems are very difficult, and thanks to the efforts of many scientists and mathematicians, there are nowadays highly sophisticated mathematical techniques for treating them.

However, a simple question remains: Are there any examples of solvable N-body problems beyond $N = 2$? The answer is yes, but it depends on the type of interactions between the bodies. In this context the seminal work of F. Calogero [3] represents a major achievement, because it describes a maximally integrable system for N particles [1] with an interaction potential depending quadratically on the inverse of the distance among two particles. Since its appearance the Calogero Model has been object of intensive research and many results have enlarged along the years its original field of study (see for example [2] and the literature index cited there).

The present work aims to be an introduction to Calogero models considering a simple problem with just three particles. Based on the work by Correa, Lechtenfeld and Plyushchay [1], we intend to find a Casimir operator which includes the given interaction potential, it is symmetric under particle permutations, and very important, the center-of-mass (COM) and total momentum are explicitly decoupled. This Casimir operator for $N = 3$ particles will be expressed using a special type of operator ordering, namely, the Weyl symmetric ordering [4], and given in terms of a certain class of algebraic operators related in a simple way to the usual position and momentum operators. Additionally, the final expression should contain the already known $\mathfrak{sl}(2)$ Casimir operator for $N = 2$ particles [1, 2], and it must account for both free-interaction and classical approximation (Poisson brackets) cases.

Concerning the structure of this work, we start with a brief description of the Calogero model, with focus on the properties to be used later, then we will introduce the Weyl ordering and the operators B_{kl} along with its properties, to proceed with the central idea behind the calculations for determining this Casimir operator. The detailed steps, proofs and algebraic calculations are included in the appendix at the end of this work.

1.1 Elements of Calogero Models [1, 2]

In what follows we just give a short description of the Calogero model, with focus on the most relevant parts for the present work. In particular, our main reference is [1], but a detailed review on the subject can be found in the work by Polychronakos [2].

For the mathematical description we use typical position and momentum coordinates x^i, p_i, respectively, satisfying

$$\left[x^j, p_k\right] = i\delta^j_k;\ p_k = -i\frac{\partial}{\partial x^k} = -i\partial_k \ , \tag{1.1}$$

the indices j, k running from one to N; being the space flat for this problem, only lower indices will be employed to denote the components of vectors, tensors, operators, etc.
The Hamiltonian function which describes this N-particle model is given by [1, 2],

$$H = \frac{1}{2}\sum_i p_i^2 + \sum_{i<j}\frac{g(g-1)}{\left(x^i - x^j\right)^2} \ , \tag{1.2}$$

where $\hbar = 1$ in suitable units and all masses are normalized to the unity. The parameter g is the coupling of the interaction, being evident its vanishing when $g = 0$ or $g = 1$. Moreover, since the coupling appears in the form $g(g-1)$, the Hamiltonian does not change when $g \to 1-g$. According to the literature, there will be N constants of motion I_k, which can be defined in terms of Dunkl operators [1]

$$\pi_i = p_i + \imath \sum_{j(j\neq i)}\frac{g}{x^i - x^j}s_{ij} \iff \mathcal{D}_i = \partial_i - \sum_{j(j\neq i)}\frac{g}{x^i - x^j}s_{ij} \ , \tag{1.3}$$

where s_{ij} is the 2-particle permutation operator satisfying:

$$s_{ij}x^i = x^j s_{ij};\ s_{ij}\partial_i = \partial_j s_{ij};\ s_{ij}^2 = 1 \ . \tag{1.4}$$

The definition of I_k is then

$$I_k = res\left(\sum_\mu^N \pi_\mu^k\right) \ , \tag{1.5}$$

being $res\,(A)$ the restriction of a given operator A to the subspace of states totally symmetric under two-particle exchange. Those Dunkl operators satisfy a non-trivial relation (though easy to proof by direct calculation, see Section A.2 of the appendix)

$$[\pi_i, \pi_j] = 0\ \forall i, j = 1, \ldots, N \ , \tag{1.6}$$

which generates the commutation of the constants of motion I_k between themselves:

$$[I_k, I_l] = 0\ \forall\ k, l = 1 \ldots N \ . \tag{1.7}$$

From their definition, it is easy to calculate some integrals of motion I_k; in particular, the Hamiltonian H in (1.2) corresponds to I_2, so we can immediately write

$$[I_k, H] = 0 \ . \tag{1.8}$$

This is a very important result, and we will make use of it when postulating the new operators for the Casimir candidate; technically it is said that "*the I_k form N involutive constants of motion*"[1].

Considering the operators

$$D = \frac{1}{2}\sum_i \left(x^i p_i + p_i x^i\right) \ , \tag{1.9}$$

$$K = \frac{1}{2}\sum_i \left(x^i\right)^2 \ , \tag{1.10}$$

together with the Hamiltonian (1.2) they form an $\mathfrak{sl}(2)$ algebra

$$\begin{aligned} [D, H] &= 2iH \\ [D, K] &= -2iK \ , \\ [K, H] &= iD \end{aligned} \tag{1.11}$$

which possesses a known Casimir element. Combining the operators D and K with the constants of motion I_k, one can show we are in presence of a Witt algebra:

$$[D, I_k] = \imath k I_k \ , \tag{1.12}$$

$$[K, I_m] =: \imath l J_m \ . \tag{1.13}$$

$$\Longrightarrow \imath[J_k, J_l] = (k-l)J_{k+l-2} \ . \tag{1.14}$$

Before proceeding to analyze the tools required for the determination of the Casimir operator we give a general description of the program for the next sections:

* We will start by defining the operators to be employed in the Casimir operator Ansatz. In particular, we will choose a suitable basis for the case $N = 3$ particles and later all their commutation relations will be determined. At some point we will explore the possibility of decoupling the center-of-mass (COM) and total momentum of the system, because the main goal is to find a Casimir operator as simple as possible, in which center-of-mass and total momentum terms are decoupled;

* In the next step, we formulate an Ansatz for the Casimir operator in the free case, based on what we already know from standard quantum and classical mechanics. Since there will be many unknown coefficients in the proposed operator, we will have to find a system of equations relating those variables, this being achieved by imposing the vanishing of the commutators with the basis operators; this is the main part of this work, and here many algebraic difficulties will appear. Later we will also check that this operator not only give a quantum solution, but a classical one (using Poisson brackets), too;

* After having found a free Casimir operator, we will switch-on the interaction g in the Hamiltonian function. The way in which this interaction affects the basis operators and their commutation relations will be analyzed, and considering the new conditions we will search for the modified Casimir operator.

For the sake of simplicity only the main results will be shown, because many algebraic calculations were needed. In the appendix (Section A) the reader can find many algebraic details concerning those results, explicit derivations and the underlying ideas behind the results.

2 Operators B_{kl}

2.1 Definitions

Due to the role it will play in latter sections, we start by defining **Weyl ordering** of operators [4]: Given k operators A_k, their Weyl ordering product reads

$$W(A_1, A_2, \ldots, A_k) := (A_1|A_2|\ldots|A_k) = \frac{1}{k!}\sum_{\sigma} A_{\sigma(1)} \ldots A_{\sigma(k)} \ , \tag{2.1}$$

with "σ" one of the $k!$ permutations between k elements. Another definition of this special ordering uses an exponential function

$$W(A_1, \ldots, A_n) = \frac{\partial}{\partial \alpha_1} \cdots \frac{\partial}{\partial \alpha_n} e^{\alpha_1 A_1 + \cdots + \alpha_n B_n}\bigg|_{\alpha_1 = \cdots = \alpha_n = 0} \ , \tag{2.2}$$

but we will employ the first definition, owing to its simplicity when expanding the algebraic expressions containing Weyl-ordered products. Here are some practical examples of this new operator ordering:

$$(A|B) = \frac{1}{2}(AB + BA) \ , \tag{2.3}$$

$$(A|B|B) = \left[\frac{3!}{2!}\right]^{-1} \left(AB^2 + BAB + B^2A\right) \ . \tag{2.4}$$

It is highly important to note the difference between (2.4) and the following product:

$$(A|B^2) = \frac{1}{2}\left(AB^2 + B^2A\right) \neq (A|B|B) \ . \tag{2.5}$$

This difference will play a crucial role when determining the system of equations governing the equations associated to the coefficients in the Casimir operator.

Returning to Polychronakos [2], the idea is to build operators which are symmetric under particle permutations; clearly, the position and momenta coordinates do not satisfy this requirement, but with a suitable linear combination of those operators the permutation symmetry is preserved. For example, one can build the operators [2]

$$I_{kl} = \sum_{\mu}^{N} : x_\mu^k p_\mu^l : \ , \tag{2.6}$$

with :: some definite operator ordering (normal ordering, etc); they are invariant under particle permutations. Defining

$$I(k, q) = \sum_{m,n=0}^{\infty} \frac{k^m q^n}{m!n!} I_{m,n} \ , \tag{2.7}$$

and assuming Weyl ordering for the $I_{m,n}$, it is found

$$[I(k, q), I(k', q')] = 2\imath \sin\frac{(kq' - k'q)}{2} \cdot I(k + k', q + q') \ , \tag{2.8}$$

being the lowest order in $\hbar$

$$[I_{m,n}, I_{m',n'}] = \imath\hbar(mn' - nm')I_{m+m'-1,n+n'-1} + \mathcal{O}(\hbar^3) \ . \tag{2.9}$$

This last expression is an example of the so-called $W_{1+\infty}$-algebras. How can we visualize this algebra? With the help of the Runge-Lenz vector associated to planetary motion, or particle in a Coulomb field [5] :

$$H = \frac{p^2}{2m} - \frac{\mu}{r} \ . \tag{2.10}$$

The additional symmetry of this Hamiltonian is the Runge-Lenz vector

$$\vec{R} = \vec{L} \times \vec{p} + \mu\frac{\vec{r}}{r} \ . \tag{2.11}$$

Calculating the commutators between the components of the angular momentum $\vec{L}$ and Runge-Lenz vector, the results describe a non-linear algebra

$$[L_i, L_j] = \epsilon_{ijk} L_k \ , \tag{2.12}$$

$$[R_i, L_j] = \epsilon_{ijk} R_k \ , \tag{2.13}$$

$$[R_i, R_j] = \epsilon_{ijk} \left(\lambda\vec{L}^2 - 2H\right) L_k \ , \tag{2.14}$$

because the last term in the last commutator is HL_k, which clearly is not linear. This structure, namely, a term with just one operator plus additional terms which are product of operators is an example of the so-called W-algebra. Another example is given by the Virasoro algebra in String Theory; defining $L_n := z^n\partial_z$

$$\begin{aligned} [C, L_m] &= 0 \ , \\ [L_n, L_m] &= (m-n)L_{m+n} + \frac{1}{12}\left(m^3 - m\right)\delta_{m,-n} \ , \end{aligned} \tag{2.15}$$

in this case with commutator of the form some operator L_j plus a constant term. A generalization is the W_3 associative algebra [6]:

$$\begin{aligned} [1, L_n] &= [1, W_m] = 0 \ , \\ [L_n, L_m] &= (m-n)L_{m+n} + \frac{1}{12}\left(m^3 - m\right)\delta_{m,-n} \ , \\ [L_n, W_m] &= (2n-m)W_{n+m} \ , \end{aligned} \tag{2.16}$$

$$\begin{aligned} [W_m, W_n] = & \frac{c}{360}m(m^2-1)(m^2-4)\delta_{m+n,0} + \\ & + (m-n)\left(\frac{1}{15}(m+n+3)(m+n+2) - \frac{1}{6}(m+2)(n+2)L_{m+n}\right) + \\ & + \beta\sum_n (L_{m-n}L_n) - \frac{3}{10}(m+3)(m+2)L_m \ . \end{aligned} \tag{2.17}$$

Motivated by the previous ideas, the Casimir operator will be expressed in terms of the following B_{kl}-operators, considering Weyl ordering:

$$B_{kl} = \frac{1}{2}\sum_{\alpha=1}^{N}\left(x_\alpha^k p_\alpha^l + p_\alpha^l x_\alpha^k\right) \ , \tag{2.18}$$

being "N" the number of particles and $[x_k, p_l] = i\delta_{kl}$ ($\hbar = 1$). Moreover, by construction they will satisfy a W-algebra. A few important conventions before we continue:

* The operator x^k is considered as one operator and not as the product of k x-operators; the same applies to p^l. With this in mind, we can also define

$$B_{kl} = \sum_{\mu=1}^{N}\left(x_\mu^k | p_\mu^l\right) \ . \tag{2.19}$$

* Later we will see expressions like $(B_{kl}|B_{mn}|B_{op})$. In those cases we always mean Weyl ordering expansion for three operators B_{kl}, B_{mn} and B_{op}:

$$(B_{kl}|B_{mn}|B_{op}) = \frac{1}{3!}\left[B_{kl}B_{mn}B_{op} + B_{kl}B_{op}B_{mn} + B_{mn}B_{kl}B_{op} + \right.$$
$$\left. + B_{mn}B_{op}B_{kl} + B_{op}B_{kl}B_{mn} + B_{op}B_{mn}B_{kl}\right] \ . \tag{2.20}$$

These operators can be related to the work [1], in which D, K, I_m and J_n are used (eqs. (1.12) to (1.14)):

$$D \to B_{11}; \ K \to B_{20}; \ I_m \to B_{0m}; \ J_n \to B_{1,n-1} \ . \tag{2.21}$$

No matter if we are in the free or interaction case, the relations between D, K, I_m and J_n must be reproduced when using the operators B_{kl}; this is a simple test for verifying that our proposed method is consistent.

2.2 Commutator $[B_{kl}, B_{mn}]$

Since the Casimir operator will be expressed in terms of the operators B_{kl}, and by construction it commutes with any operator of the given algebra, the commutation relations for these operators must be calculated. One can do this directly, using the definition of each operator B_{kl} in terms of x_α^k and p_α^l, but it sounds more reasonable trying to find a general formula and then to apply it in the case of the basis operators, for example. How to find such a general formula? By direct calculation:

i) We must get a general expression for $[x^j, p]$ and $[x, p^k]$, $j, k \in \mathbb{N}$;

ii) Then we proceed to find $[x^j, p^k]$ in terms of $x^\alpha p^\beta$ and/or $p^\beta x^\alpha$ (here the order is important!!);

iii) Using the definition of B_{kl}, namely

$$B_{kl} = \frac{1}{2} \sum_{\alpha=1}^{N} \left(x_\alpha^k p_\alpha^l + p_\alpha^l x_\alpha^k \right) \ , \tag{2.22}$$

we calculate the commutator $[B_{kl}, B_{mn}]$;

4) By expanding in powers of $x^j p^k$ (or $p^k x^j$) up to some order (which is depending on the number of particles of the system), and collecting similar terms we will try to reorder the expressions in terms of suitable combinations $B_{k+m-\alpha;l+n-\alpha}$ with $\alpha = 1, \ldots, 3$.

The calculations, though being trivial in most of the cases, are lengthy and cumbersome; just a view to the formal expansion of the commutator $[B_{kl}, B_{mn}]$ shows the algebraic complexity:

$$[B_{kl}, B_{mn}] = \frac{1}{4} \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} \Big(-x_\alpha^k \left[x_\beta^m, p_\alpha^l \right] p_\beta^n + x_\beta^m \left[x_\alpha^k, p_\beta^n \right] p_\alpha^l +$$

$$- x_\alpha^k p_\beta^n \left[x_\beta^m, p_\alpha^l \right] + \left[x_\alpha^k, p_\beta^n \right] x_\beta^m p_\alpha^l + p_\alpha^l x_\beta^m \left[x_\alpha^k, p_\beta^n \right] +$$

$$- \left[x_\beta^m, p_\alpha^l \right] p_\beta^n x_\alpha^k + p_\alpha^l \left[x_\alpha^k, p_\beta^n \right] x_\beta^m - p_\beta^n \left[x_\beta^m, p_\alpha^l \right] x_\alpha^k \Big) \ . \tag{2.23}$$

By this reason the description of the main steps is included in the appendix A.1. In the end, we obtain the following results:

$$\Big[B_{kl}, B_{mn} \Big] = iC_{klmn}^1 \mathbf{B_{k+m-1;l+n-1}} + iC_{klmn}^3 \mathbf{B_{k+m-3;l+n-3}} + \mathcal{O}\left(B_{k+m-5;l+n-5} \right) \ , \tag{2.24}$$

$$C_{klmn}^1 = kn - lm \ , \tag{2.25}$$

$$C_{klmn}^3 = \frac{1}{12} \Big\{ k(k-1)n(n-1) \left[(k-2+3m)(n-2+3l) - 3lm \right] +$$
$$\text{-}l(l-1)m(m-1) \left[(l-2+3n)(m-2+3k) - 3kn \right] \Big\} \ . \tag{2.26}$$

With the above found formulas is the first task accomplished. We recall, our goal is to find a Casimir operator using the operators B_{kl}. Is there any basis for these operators? The answer is yes, and this basis will depend on the number of particles. For two particles, the basis is:

$$\begin{aligned}
B_{00} &= \sum_{i=1}^{2} = 2 \ , \\
B_{10} &= \sum_{i=1}^{2} x_i = x_1 + x_2 \ , \\
B_{01} &= \sum_{i=1}^{2} p_i = p_1 + p_2 \ , \\
B_{20} &= \sum_{i=1}^{2} x_i^2 = x_1^2 + x_2^2 \ , \\
B_{02} &= \sum_{i=1}^{2} p_i^2 = p_1^2 + p_2^2 \ , \\
B_{11} &= \frac{1}{2}\sum_{i=1}^{2} x_i p_i + p_i x_i = \frac{1}{2}\left(x_1 p_1 + p_1 x_1 + x_2 p_2 + p_2 x_2\right) \ .
\end{aligned} \tag{2.27}$$

How many independent B_{kl} operators for $N = 3$ particles are there? The rule states that all operators B_{kl} with $k + l \leq N$ are linear independent. Then, the basis for our problem reads:

$$\left\{ \begin{array}{llll} \text{Level 0: } B_{00} & & & \\ \text{Level 1: } B_{10} & B_{01} & & \\ \text{Level 2: } B_{20} & B_{02} & B_{11} & \\ \text{Level 3: } B_{30} & B_{21} & B_{12} & B_{03} \end{array} \right\} \ . \tag{2.28}$$

Though $B_{00} = 3$ is just a number, we count it as part of the basis, because some linear dependent operators could have constants terms and those ones are represented by some multiple of B_{00}. The operators B_{10} and B_{01} are the center-of-mass and total momentum, respectively; in a first Ansatz for the Casimir operator they could appear, but since the main idea of this project is to decouple the center-of-mass operators, necessarily a transformation must contain a combination of B_{10} and B_{01} to achieve its "elimination" and so to simplify the remaining algebraic calculations.

It is important to mention that the basis operators for $N = 2$ and $N = 3$ particles are already Weyl-ordered products; this is evident for operators of the form $B_{k;0}$ and $B_{0;l}$, with $k, l = 0 \ldots N$, also for B_{11}. For the case B_{21} one must expand the Weyl ordering of $x_\alpha^2 p_\alpha$ as follows

$$W(x_\alpha^2 p_\alpha) = \left(\frac{3!}{2!}\right)^{-1} \left[x_\alpha^2 p_\alpha + x_\alpha p_\alpha x_\alpha + p_\alpha x_\alpha^2\right] \ , \tag{2.29}$$

and by rewriting $x_\alpha p_\alpha x_\alpha$ in terms of $x_\alpha^2 p_\alpha$ and $p_\alpha x_\alpha^2$ one gets $W(B_{21}) = B_{21}$, the technique being the same for the operator B_{12}. With the same argument, this interesting property of the basis operators is false for $N \geq 4$.

Having defined a basis of operators, the next step is to calculate their commutation relations using the formulas (2.24), (2.25) and (2.26); the results are shown in the Table 1.

$\imath^{-1}[B_{kl}, B_{mn}]$	B_{10}	B_{01}	B_{20}	B_{02}	B_{11}	B_{30}	B_{21}	B_{12}	B_{03}
B_{10}	0	N	0	$2B_{01}$	B_{10}	0	B_{20}	$2B_{11}$	$3B_{02}$
B_{01}	$-N$	0	$-2B_{10}$	0	$-B_{01}$	$-3B_{20}$	$-2B_{11}$	$-B_{02}$	0
B_{20}	0	$2B_{10}$	0	$4B_{11}$	$2B_{20}$	0	$2B_{30}$	$4B_{21}$	$6B_{12}$
B_{02}	$-2B_{01}$	0	$-4B_{11}$	0	$-2B_{02}$	$-6B_{21}$	$-4B_{12}$	$-2B_{03}$	0
B_{11}	$-B_{10}$	B_{01}	$-2B_{20}$	$2B_{02}$	0	$-3B_{30}$	$-B_{21}$	B_{12}	$3B_{03}$
B_{30}	0	$3B_{20}$	0	$6B_{21}$	$3B_{30}$	0	$3B_{40}$	$6B_{31}$	$+9B_{22}$ $+3B_{00}$
B_{21}	$-B_{20}$	$2B_{11}$	$-2B_{30}$	$4B_{12}$	B_{21}	$-3B_{40}$	0	$+3B_{22}$ $+2B_{00}$	$6B_{13}$
B_{12}	$-2B_{11}$	B_{02}	$-4B_{21}$	$2B_{03}$	$-B_{12}$	$-6B_{31}$	$-3B_{22}$ $-2B_{00}$	0	$3B_{04}$
B_{03}	$-3B_{02}$	0	$-6B_{12}$	0	$-3B_{03}$	$-9B_{22}$ $-3B_{00}$	$-6B_{13}$	$-3B_{04}$	0

Table 1: Commutators $[B_{kl}, B_{mn}]$.

A brief analysis of them shows:

* The commutators between B_{10} and B_{01} yield either $B_{00} = N$ or zero;

* There is a subsector generated by B_{20}, B_{02} and B_{11}, which is closed and resembles strongly the angular momentum algebra from Quantum Mechanics;

* Not all the commutators generate linear independent operators. For example

$$[B_{30}, B_{03}] = 9iB_{22} + 3iB_{00} \tag{2.30}$$

or

$$[B_{30}, B_{12}] = 6iB_{31} \ . \tag{2.31}$$

So, there arises the extra problem of expressing the operators

$$B_{40}, B_{04}, B_{31}, B_{13}, B_{22} \tag{2.32}$$

using the basis (2.28). How can this be achieved? By means of Weyl ordering, as in Section 2.3 is carefully analyzed. For example:

$$B_{22} = \frac{2}{3}(21|01) + \frac{1}{6}(20|02) + \frac{2}{3}(12|10) + \frac{1}{3}(11|11) + \\ - \frac{1}{6}(20|01|01) - \frac{2}{3}(11|10|01) - \frac{1}{6}(10|10|02) + \\ + \frac{1}{6}(10|10|01|01) - \frac{B_{00}}{2} \ . \quad (2.33)$$

This operator shows the difficulties that will appear when determining the Casimir operator. For this reason, we will search for a suitable change of basis, in which the center-of-mass and total momentum will not explicitly appear and B_{22} will be of the form

$$B'_{22} \sim \alpha(20'|02') + \beta(11'|11') + \gamma, \quad (2.34)$$

with $\alpha, \beta, \gamma \in \mathbb{R}$.

2.3 B_{22} - General Case

Considering COM, the operator B_{22} can be expressed as follows according to the number N of particles:

$$B_{22} = A\,(21|01) + B\,(20|02) + C\,(12|10) + D\,(11|11) + E\,(20|10|10) + F\,(11|10|01) + \\ + G\,(10|10|02) + H\,(10|10|01|01) + J, \;\; N = 3 \ , \quad (2.35)$$

$$B_{22} = B\,(20|02) + D\,(11|11) + E\,(20|10|10) + F\,(11|10|01) + \\ + G\,(02|10|10) + H\,(10|10|01|01) + J, \;\; N = 2 \ . \quad (2.36)$$

For $N \geq 4$ this operator is part of the basis, and there is no possible decomposition in terms of lower order B_{kl} operators. In both cases $N = 2$ and $N = 3$ the technique to be applied is very simple: finding expressions for each operator by choosing the ordering $x^j p^k$, then we insert in the respective formula and compare coefficients in the given expression for B_{22}:

$$\begin{aligned} B_{22} &= \frac{1}{2}\sum_{\alpha=1}^{N} \{x_\alpha^2, p_\alpha^2\} = \frac{1}{2}\sum_{\alpha=1}^{N} \left(2x_\alpha^2 p_\alpha^2 - \left[x_\alpha^2, p_\alpha^2\right]\right) \\ &= \sum_{\alpha}^{N} x_\alpha^2 p_\alpha^2 - \frac{1}{2}\sum_{\alpha}^{N} 2i \cdot (x_\alpha p_\alpha + p_\alpha x_\alpha) \\ &= \sum_{\alpha}^{N} x_\alpha^2 p_\alpha^2 - \frac{1}{2}\sum_{\alpha}^{N} 2i \cdot (2x_\alpha p_\alpha - i) \\ &= \sum_{\alpha=1}^{N} x_\alpha^2 p_\alpha^2 - 2i\sum_{\alpha=1}^{N} x_\alpha p_\alpha + Ni^2 \ . \end{aligned} \quad (2.37)$$

Similarly:

$$B_{11} = \frac{1}{2}\sum_{\alpha=1}^{N}\{x_\alpha, p_\alpha\} = \frac{1}{2}\sum_{\alpha=1}^{N}(2x_\alpha p_\alpha - [x_\alpha, p_\alpha]) = \sum_{\alpha}^{N} x_\alpha p_\alpha - \frac{Ni}{2} \ , \tag{2.38}$$

$$B_{21} = \frac{1}{2}\sum_{\alpha=1}^{N}\left\{x_\alpha^2, p_\alpha\right\} = \sum_{\alpha=1}^{N} x_\alpha^2 p_\alpha - \frac{1}{2}\left[x_\alpha^2, p_\alpha\right] = \sum_{\alpha}^{N} x_\alpha^2 p_\alpha - i\sum_{\alpha}^{N} x_\alpha \ , \tag{2.39}$$

$$B_{12} = \frac{1}{2}\sum_{\alpha=1}^{N}\left\{x_\alpha, p_\alpha^2\right\} = \sum_{\alpha=1}^{N} x_\alpha p_\alpha^2 - \frac{1}{2}\left[x_\alpha, p_\alpha^2\right] = \sum_{\alpha}^{N} x_\alpha p_\alpha^2 - i\sum_{\alpha}^{N} p_\alpha \ . \tag{2.40}$$

We describe the general method in two cases: (21|01) and (10|10|01|01). In all other cases the procedure is the same, and for this reason we only indicate the results.

A) (**21**|**01**). Reordering the Weyl-ordered product:

$$\begin{aligned}(21|01) &= \frac{1}{2}\{(21),(01)\} \\ &= (21)\cdot(01) - \frac{1}{2}[(21),(01)] \\ &= (21)\cdot(01) - i(11) \ .\end{aligned} \tag{2.41}$$

Inserting (2.39) and (2.38) in the last equation we get:

$$\begin{aligned}(21|01) &= \left(\sum_{\alpha}^{N} x_\alpha^2 p_\alpha - i\sum_{\alpha}^{N} x_\alpha\right)\sum_{\beta}^{N} p_\beta - i\sum_{\alpha}^{N} x_\alpha p_\alpha + \frac{Ni^2}{2} \\ &= \sum_{\alpha}^{N} x_\alpha^2 p_\alpha^2 + \sum_{\alpha\neq\beta}^{N} x_\alpha^2 p_\alpha p_\beta - 2i\sum_{\alpha}^{N} x_\alpha p_\alpha - i\sum_{\alpha\neq\beta}^{N} x_\alpha p_\beta + \frac{Ni^2}{2} \ .\end{aligned} \tag{2.42}$$

B) (**10**|**10**|**01**|**01**). Applying the formula

$$(A|B|C|D) = \frac{1}{4}\left[A(B|C|D) + B(A|C|D) + C(A|B|D) + D(A|B|C)\right] \ , \tag{2.43}$$

the expression to be reduced is

$$(10|10|01|01) = \frac{1}{2}\cdot(10)\cdot(10|01|01) + \frac{1}{2}\cdot(01)\cdot(10|10|01) \ . \tag{2.44}$$

For the first term:

$$\begin{aligned}3(10|01|01) &= (10)\cdot(01)^2 + 2\cdot(01)\cdot(10|01) \\ 3(10|01|01) &= (10)\cdot(01)^2 + 2\cdot(10|01)\cdot(01) - 2Ni\cdot(01) \\ 3(10|01|01) &= 3\cdot(10)\cdot(01)^2 - 3Ni\cdot(01) \\ (10|01|01) &= (10)\cdot(01)^2 - Ni\cdot(01) \ ,\end{aligned} \tag{2.45}$$

$$\therefore 10 \cdot (10|01|01) = (10)^2 \cdot (01)^2 - Ni \cdot (10) \cdot (01) \ . \tag{2.46}$$

In the second term of (2.44) we change the order of the factors:

$$\begin{aligned}
(01) \cdot (10|10|01) &= (10|10|01) \cdot (01) + [01, (10|10|01)] \\
&= (10|10|01) \cdot (01) - 2Ni \cdot (10|01) \\
&= (10|10|01) \cdot (01) - 2Ni \cdot (10) \cdot (01) + N^2 i^2 \ .
\end{aligned} \tag{2.47}$$

Transforming (10|10|01):

$$\begin{aligned}
3(10|10|01) &= 2 \cdot (10) \cdot (10|01) + 01 \cdot (10|10) \\
3(10|10|01) &= 2 \cdot (10)^2 \cdot (01) - Ni \cdot (10) - 2Ni \cdot (10) + (10)^2 \cdot (01) \\
3(10|10|01) &= 3 \cdot (10)^2 \cdot (01) - 3Ni \cdot (10) \\
(10|10|01) &= (10)^2 \cdot (01) - Ni \cdot (10) \ .
\end{aligned} \tag{2.48}$$

Inserting (2.48) in (2.47)

$$(01) \cdot (10|10|01) = (10)^2 \cdot (01)^2 - 3Ni \cdot (10) \cdot (01) + N^2 i^2 \ . \tag{2.49}$$

Combining (2.46) and (2.49) and inserting in (2.44), we arrive to the result

$$(10|10|10|01) = (10)^2 \cdot (01)^2 - 2Ni \cdot (10) \cdot (01) + \frac{N^2 i^2}{2} \ . \tag{2.50}$$

Finally, after replacement of B_{10} and B_{01}, the expression for (10|10|01|01) reads

$$\begin{aligned}
(10|10|01|01) = &\sum_{\alpha}^{N} x_\alpha^2 p_\alpha^2 + 2\sum_{\alpha\neq\beta}^{N} x_\alpha^2 p_\alpha p_\beta + 2\sum_{\alpha\neq\beta}^{N} x_\alpha x_\beta p_\beta^2 + \sum_{\alpha\neq\beta}^{N} x_\alpha^2 p_\beta^2 + \sum_{\alpha\neq\beta\neq\gamma}^{N} x_\alpha^2 p_\beta p_\gamma + \\
&+ 2\sum_{\alpha\neq\beta}^{N} x_\alpha x_\beta p_\beta p_\alpha + 4\sum_{\alpha\neq\beta\neq\gamma}^{N} x_\alpha x_\beta p_\beta p_\gamma + \sum_{\alpha\neq\beta\neq\gamma}^{N} x_\alpha x_\beta p_\gamma^2 + \sum_{\alpha\neq\beta\neq\gamma\neq\tau}^{N} x_\alpha x_\beta p_\gamma p_\tau \\
&- 2iN\sum_{\alpha}^{N} x_\alpha p_\alpha - 2Ni\sum_{\alpha\neq\beta}^{N} x_\alpha p_\beta + \frac{N^2 i^2}{2} \ .
\end{aligned} \tag{2.51}$$

When calculating the sums, one must expand carefully, because there will be sums over two, three and four different indices, and all possible permutations must be included. For example

$$\begin{aligned}
\sum_{\alpha,\beta,\gamma} f(\alpha,\beta,\gamma) = &\sum_{\alpha=\beta=\gamma} f(\alpha,\alpha,\alpha) + \\
&\sum_{\alpha=\beta\neq\gamma} f(\alpha,\alpha,\gamma) + \sum_{\alpha=\gamma\neq\beta} f(\alpha,\beta,\alpha) + \sum_{\alpha\neq\beta=\gamma} f(\alpha,\beta,\beta) + \\
&\sum_{\alpha\neq\beta\neq\gamma} f(\alpha,\beta,\gamma) \ ,
\end{aligned} \tag{2.52}$$

$$\sum_{\alpha,\beta,\gamma,\delta} f(\alpha,\beta,\gamma,\delta) = \sum_{\alpha=\beta=\gamma=\delta} f(\alpha,\alpha,\alpha,\alpha) + \sum_{\alpha=\beta=\gamma\neq\delta} f(\alpha,\alpha,\alpha,\delta) +$$
$$\sum_{\alpha=\beta=\delta\neq\gamma} f(\alpha,\alpha,\gamma,\alpha) + \sum_{\alpha=\gamma=\delta\neq\beta} f(\alpha,\beta,\alpha,\alpha) + \sum_{\beta=\gamma=\delta\neq\alpha} f(\alpha,\beta,\beta,\beta) +$$
$$\sum_{(\alpha=\beta)\neq(\gamma=\delta)} f(\alpha,\alpha,\gamma,\gamma) + \sum_{(\alpha=\beta)\neq(\gamma\neq\delta)} f(\alpha,\alpha,\gamma,\delta) + \sum_{(\alpha=\gamma)\neq(\beta=\delta)} f(\alpha,\beta,\alpha,\beta) +$$
$$\sum_{(\alpha=\gamma)\neq(\beta\neq\delta)} f(\alpha,\beta,\alpha,\delta) + \sum_{(\alpha=\delta)\neq(\beta=\gamma)} f(\alpha,\beta,\beta,\alpha) + \sum_{(\alpha=\delta)\neq(\beta\neq\gamma)} f(\alpha,\beta,\gamma,\alpha) +$$
$$\sum_{(\alpha\neq\delta)\neq(\beta=\gamma)} f(\alpha,\beta,\beta,\delta) + \sum_{(\alpha\neq\gamma)\neq(\beta=\delta)} f(\alpha,\beta,\gamma,\beta) + \sum_{(\alpha\neq\beta)\neq(\gamma=\delta)} f(\alpha,\beta,\gamma,\gamma) +$$
$$\sum_{\alpha\neq\beta\neq\gamma\neq\delta} f(\alpha,\beta,\gamma,\delta) \ . \quad (2.53)$$

Moreover, some terms will not appear in the case $N = 3$ particles; the last term of (2.53)

$$\sum_{\alpha\neq\beta\neq\gamma\neq\delta}^{N} f(\alpha,\beta,\gamma,\delta) \ , \quad (2.54)$$

will vanish, because it is not possible to find four different particles in a problem with just three of them.

Applying the same procedure to the rest of the terms in B_{22}, these are the final results:

$$(21|01) = \sum_{\alpha}^{N} x_\alpha^2 p_\alpha^2 + \sum_{\alpha\neq\beta}^{N} x_\alpha^2 p_\alpha p_\beta - 2i \sum_{\alpha}^{N} x_\alpha p_\alpha - i \sum_{\alpha\neq\beta}^{N} x_\alpha p_\beta + \frac{Ni^2}{2} \ , \quad (2.55)$$

$$(20|02) = \sum_{\alpha}^{N} x_\alpha^2 p_\alpha^2 + \sum_{\alpha\neq\beta}^{N} x_\alpha^2 p_\beta^2 - 2i \sum_{\alpha}^{N} x_\alpha p_\alpha + Ni^2 \ , \quad (2.56)$$

$$(12|10) = \sum_{\alpha}^{N} x_\alpha^2 p_\alpha^2 + \sum_{\alpha\neq\beta}^{N} x_\alpha x_\beta p_\beta^2 - 2i \sum_{\alpha}^{N} x_\alpha p_\alpha - i \sum_{\alpha\neq\beta}^{N} x_\alpha p_\beta + \frac{Ni^2}{2} \ , \quad (2.57)$$

$$(11|11) = \sum_{\alpha}^{N} x_\alpha^2 p_\alpha^2 + \sum_{\alpha\neq\beta}^{N} x_\alpha x_\beta p_\beta p_\alpha - i(N+1) \sum_{\alpha}^{N} x_\alpha p_\alpha + \frac{N^2 i^2}{4} \ , \quad (2.58)$$

$$(20|01|01) = \sum_{\alpha}^{N} x_\alpha^2 p_\alpha^2 + 2\sum_{\alpha\neq\beta}^{N} x_\alpha^2 p_\alpha p_\beta + \sum_{\alpha\neq\beta}^{N} x_\alpha^2 p_\beta^2 + \sum_{\alpha\neq\beta\neq\gamma}^{N} x_\alpha^2 p_\beta p_\gamma +$$
$$- 2i \sum_{\alpha}^{N} x_\alpha p_\alpha - 2i \sum_{\alpha\neq\beta}^{N} x_\alpha p_\beta + \frac{2Ni^2}{3} \ , \quad (2.59)$$

$$(11|10|01) = \sum_{\alpha}^{N} x_\alpha^2 p_\alpha^2 + \sum_{\alpha\neq\beta}^{N} x_\alpha^2 p_\alpha p_\beta + \sum_{\alpha\neq\beta}^{N} x_\alpha x_\alpha p_\beta p_\alpha + \sum_{\alpha\neq\beta}^{N} x_\alpha x_\beta p_\beta^2 + \sum_{\alpha\neq\beta\neq\gamma}^{N} x_\alpha x_\beta p_\beta p_\gamma +$$
$$- i(N+1) \sum_{\alpha}^{N} x_\alpha p_\alpha - i\left(\frac{N}{2}+1\right) \sum_{\alpha\neq\beta}^{N} x_\alpha p_\beta + \frac{Ni^2}{12}(3N+2) \ , \quad (2.60)$$

$$(02|10|10) = \sum_{\alpha}^{N} x_\alpha^2 p_\alpha^2 + \sum_{\alpha\neq\beta}^{N} x_\alpha^2 p_\beta^2 + 2\sum_{\alpha\neq\beta}^{N} x_\alpha x_\beta p_\beta^2 + \sum_{\alpha\neq\beta\neq\gamma}^{N} x_\alpha x_\beta p_\gamma^2 + \\ - 2i\sum_{\alpha}^{N} x_\alpha p_\alpha - 2i\sum_{\alpha\neq\beta}^{N} x_\alpha p_\beta + \frac{2Ni^2}{3} \ , \quad (2.61)$$

$$(10|10|01|01) = \sum_{\alpha}^{N} x_\alpha^2 p_\alpha^2 + 2\sum_{\alpha\neq\beta}^{N} x_\alpha^2 p_\alpha p_\beta + 2\sum_{\alpha\neq\beta}^{N} x_\alpha x_\beta p_\beta^2 + \sum_{\alpha\neq\beta}^{N} x_\alpha^2 p_\beta^2 + \sum_{\alpha\neq\beta\neq\gamma}^{N} x_\alpha^2 p_\beta p_\gamma + \\ + 2\sum_{\alpha\neq\beta}^{N} x_\alpha x_\beta p_\beta p_\alpha + 4\sum_{\alpha\neq\beta\neq\gamma}^{N} x_\alpha x_\beta p_\beta p_\gamma + \sum_{\alpha\neq\beta\neq\gamma}^{N} x_\alpha x_\beta p_\gamma^2 - 2iN\sum_{\alpha}^{N} x_\alpha p_\alpha - 2Ni\sum_{\alpha\neq\beta}^{N} x_\alpha p_\beta + \\ + \frac{N^2 i^2}{2} \ . \quad (2.62)$$

2.3.1 B_{22} - General Case with $N = 3$

Considering the constraint $N = 3$, the set of independent operators corresponds to B_{10}, B_{01}, B_{11}, B_{20}, B_{02}, B_{30}, B_{21}, B_{12}, B_{03}. The Ansatz was

$$B_{22} = A\,(21|01) + B\,(20|02) + C\,(12|10) + D\,(11|11) + E\,(20|01|01) + F\,(11|10|01) + \\ + G\,(10|10|02) + H\,(10|10|01|01) + J \ . \quad (2.63)$$

Replacing with the respective expansions of the Weyl-ordered products, and comparing order by order, the following system of equations is to be solved:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & N+1 & 2 & N+1 & 2 & 2N & 0 \\ \frac{N}{2} & N & \frac{N}{2} & \frac{N^2}{4} & \frac{2N}{3} & \frac{N}{12}(3N+2) & \frac{2N}{3} & \frac{N^2}{2} & -1 \\ 1 & 0 & 1 & 0 & 2 & \frac{N}{2}+1 & 2 & 2N & 0 \\ 1 & 0 & 0 & 0 & 2 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \\ J \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ N \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} . \quad (2.64)$$

There is a unique solution for the coefficients, given by

$$A = \frac{2}{3};\ B = \frac{1}{6};\ C = \frac{2}{3};\ D = \frac{1}{3};\ E = -\frac{1}{6};\ F = -\frac{2}{3};\ G = -\frac{1}{6};\ H = \frac{1}{6};\ J = -\frac{3}{2} = -\frac{B_{00}}{2} \ . \quad (2.65)$$

The expression for B_{22} is the following one:

$$B_{22} = \frac{2}{3}\,(21|01) + \frac{1}{6}\,(20|02) + \frac{2}{3}\,(12|10) + \frac{1}{3}\,(11|11) - \frac{1}{6}\,(20|01|01) - \frac{2}{3}\,(11|10|01) + \\ - \frac{1}{6}\,(10|10|02) + \frac{1}{6}\,(10|10|01|01) - \frac{B_{00}}{2} \ . \quad (2.66)$$

The same method is applied for the rest of the B_{kl} operators with $k + l > 3$, being the found expressions for them:

$$(40) = \frac{4}{3}(30|10) + \frac{1}{2}(20|20) - (20|10|10) + \frac{1}{6}(10|10|10|10) \ , \quad (2.67)$$

$$(04) = \frac{4}{3}(03|01) + \frac{1}{2}(02|02) - (02|01|01) + \frac{1}{6}(01|01|01|01) \ , \quad (2.68)$$

$$(31) = \frac{1}{3}(30|01) + (21|10) + \frac{1}{2}(20|11) - \frac{1}{2}(20|10|01) + - \frac{1}{2}(11|10|10) + \frac{1}{6}(10|10|10|01) \ , \quad (2.69)$$

$$(13) = \frac{1}{3}(10|03) + (12|01) + \frac{1}{2}(11|02) - \frac{1}{2}(10|02|01) - \frac{1}{2}(11|01|01) + \frac{1}{6}(10|01|01|01) \ . \quad (2.70)$$

2.4 Center-of-mass decoupling for $N = 3$ particles

The commutators between the basis operators were already determined in the previous section, and operators B_{kl} with $k + l > 3$ appeared, which must be expressed in terms of the basis operators, using Weyl ordering, a fact that complicates the expressions considerably. By this reason, the following transformation was found, which clearly simplifies the expressions and future calculations:

$$\left.\begin{array}{rcl} (10') &=& (10) \\ (01') &=& (01) \\ (00') &=& (00) = N = 3 \end{array}\right\} , \tag{2.71}$$

$$\left.\begin{array}{rcl} (20') &=& (20) - \frac{1}{3}(10|10) \\ (11') &=& (11) - \frac{1}{3}(10|01) \\ (02') &=& (02) - \frac{1}{3}(01|01) \end{array}\right\} , \tag{2.72}$$

$$\left.\begin{array}{rcl} (30') &=& (30) - (20|10) + \frac{2}{9}(10|10|10) \\ (21') &=& (21) - \frac{1}{3}(20|01) - \frac{2}{3}(11|10) + \frac{2}{9}(10|10|01) \\ (12') &=& (12) - \frac{2}{3}(11|01) - \frac{1}{3}(10|02) + \frac{2}{9}(10|01|01) \\ (03') &=& (03) - (02|01) + \frac{2}{9}(01|01|01) \end{array}\right\} . \tag{2.73}$$

With this transformation it is easy to prove

$$[(kl'), (mn)] = 0; \;\; (kl) = (10) \ldots (03); (mn) = \{(10), (01)\} \;\; , \tag{2.74}$$

i.e. the B'_{kl} operators commute by construction with B_{10} (COM) and B_{01} (total momentum). As a next step, the commutators $[B_{kl}, B_{mn}]$ must be calculated in this new basis. For example

$$\begin{aligned} [30', 03'] = [30, 03] - [30, (02|01)] + \frac{2}{9} \cdot [30, (01|01|01)] \\ - [(20|10), 03] + [(20|10), (02|01)] - \frac{2}{9} \cdot [(20|10), (01|01|01)] \\ + \frac{2}{9} \cdot [(10|10|10), 03] - \frac{2}{9} \cdot [(10|10|10), (02|01)] + \frac{4}{81} \cdot [(10|10|10), (01|01|01)] \;, \end{aligned} \tag{2.75}$$

and the result must be rewritten in terms of B'_{kl}, etc. (see appendix A.3). After some lengthy calculations, the new commutators are shown in the Table 2.

$i^{-1}[B_{kl}, B_{mn}]$	$(20')$	$(02')$	$(11')$	$(30')$	$(21')$	$(12')$	$(03')$
$(20')$	0	$4(11')$	$2(20')$	0	$2(30')$	$4(21')$	$6(12')$
$(02')$	$-4(11')$	0	$-2(02')$	$-6(21')$	$-4(12')$	$-2(03')$	0
$(11')$	$-2(20')$	$2(02')$	0	$-3(30')$	$-(21)'$	$(12')$	$3(03')$
$(30')$	0	$6(21')$	$3(30')$	0	$\frac{1}{2}(20'\vert 20')$	$(20'\vert 11')$	$-\frac{3}{2}(20'\vert 02') +3(11'\vert 11') -4$
$(21')$	$-2(30')$	$4(12')$	$(21')$	$-\frac{1}{2}(20'\vert 20')$	0	$+\frac{5}{6}(20'\vert 02') -\frac{1}{3}(11'\vert 11') +\frac{4}{3}$	$(11'\vert 02')$
$(12')$	$-4(21')$	$2(03')$	$-(12')$	$-(20'\vert 11')$	$-\frac{5}{6}(20'\vert 02') +\frac{1}{3}(11'\vert 11') -\frac{4}{3}$	0	$\frac{1}{2}(02'\vert 02')$
$(03')$	$-6(12')$	0	$-3(03')$	$+\frac{3}{2}(20'\vert 02') -3(11'\vert 11') +4$	$-(11'\vert 02')$	$-\frac{1}{2}(02'\vert 02')$	0

Table 2: Basis commutators after COM-decoupling.

From the tabulated results it can be seen, among other things, that effectively the COM and total momentum contributions were decoupled, and the commutators show results depending on B'_{20}, B'_{02}, B'_{11}, B'_{30}, B'_{21}, B'_{12} and B'_{03}. It must be emphasized that this new basis includes the operators B_{10} and B_{01}, which by construction commute with B'_{kl}. In this way, we can directly work in this new basis and postulate the Casimir candidate $\mathcal{C}$ in terms of it, being automatically guaranteed $[\mathcal{C}, B_{10}] = 0$ and $[\mathcal{C}, B_{01}] = 0$.

Another fact visible from the tables has to do with the algebra those operators obbey, being clearly a W-algebra:

* Commutators between level-2 operators, namely, B'_{20}, B'_{02} and B'_{11} yield again an operator of level 2;

* The commutator of level-3 operators B'_{30}, B'_{21}, B'_{12}, B'_{03} with level-2 operators produces a level-3 operator;

* Finally, commutators between level-3 operators generate constants and non-linear terms containing Weyl-ordered products of level-2 operators.

In summary, we have found the necessary structure for postulating a Casimir operator $\mathcal{C}$, with both decoupled COM and total momentum. We will employ consistently these B'_{kl} operators in what follows, but always bearing in mind, that their commutators contain also non-linear terms as constants, a fact that will have interesting algebraic consequences, like for example, the appearance of composed or non-pure Weyl-ordered products when dealing with $[\mathcal{C}, 30'] = 0$ and $[\mathcal{C}, 21'] = 0$. Before formulating our Ansatz for the free case, let us do a brief digression on Casimir operators in Quantum Mechanics.

2.5 Casimir Operator in Quantum Mechanics

According to standard texts on Group Theory [7, 8, 9], and Quantum Physics [10, 11, 12] for a given set of operators with well-defined commutation relations between them, a Casimir operator [13] "*is any element in the center of the universal enveloping algebra that commutes with any element of the given Lie algebra*"; this type of operators are used in connection with representation theory of semisimple Lie algebras, in particular, for irreducible representations each Casimir operator is represented by a multiple of the identity [13]. In Physics the example *par excellence* is the angular momentum algebra:

$$[J_z, J_x - iJ_y] = -2i\,(J_x - iJ_y)\ , \tag{2.76}$$

$$[J_z, J_x + iJ_y] = 2i\,(J_x + iJ_y)\ , \tag{2.77}$$

$$[J_x - iJ_y, J_x + iJ_y] = 2i^2 J_z\ . \tag{2.78}$$

In this case, the Casimir operator is given by the square of the total angular momentum $\vec{J}$:

$$J^2 = J_x^2 + J_y^2 + J_z^2\ . \tag{2.79}$$

If we consider the algebra associated to the operators $\{B_{20}, B_{02}, B_{11}\}$ in Table 2

$$[B_{11}, B_{20}] = -2iB_{20}\ , \tag{2.80}$$

$$[B_{11}, B_{02}] = 2iB_{02}\ , \tag{2.81}$$

$$[B_{20}, B_{02}] = 4iB_{11}\ , \tag{2.82}$$

they form a $\mathfrak{sl}(2)$ Lie-Algebra, whose Casimir operator will have the same form as J^2 but written in terms of the operators B_{kl} from the level 2:

$$J_x - iJ_y \to \frac{B_{20}}{2}\ , \tag{2.83}$$

$$J_x + iJ_y \to \frac{B_{02}}{2}\ , \tag{2.84}$$

$$J_z \to \frac{B_{11}}{2i}\ , \tag{2.85}$$

$$J^2 = J_x^2 + J_y^2 + J_z^2 \to \mathcal{C}_{22} = \frac{1}{4}\left[W\left(B_{20}B_{02}\right) - B_{11}^2\right]. \tag{2.86}$$

We expect that our Casimir candidate does contain this contribution, up to a constant; however, there will be other contributions considering level-3 operators and their associated non-linear operators, such as B_{22} or B_{40}, etc. Which form should the candidate exhibit? In the appendix A.4 a method is proposed to motivate and make plausible the proposed Casimir candidate, to be analyzed in the next section.

3 Casimir Operator

3.1 Free Case Ansatz

Considering the treatment for the two-particle case of the previous section (and seen in the appendix A.4), the following expression is proposed for the Casimir operator, in which the center-of-mass is already decoupled:

$$\mathcal{C} = \mathcal{C}_\alpha + \mathcal{C}_\beta + \mathcal{C}_\gamma + \mathcal{C}_\delta + \mathcal{C}_\epsilon + \mathcal{C}_\zeta \ . \tag{3.1}$$

$$\begin{aligned} \mathcal{C}_\alpha = \alpha_1\,(20'|20'|20'|02'|02'|02') + \alpha_2\,(20'|20'|11'|11'|02'|02') + \\ + \alpha_3\,(20'|11'|11'|11'|11'|02') + \alpha_4\,(11'|11'|11'|11'|11'|11') \ , \end{aligned} \tag{3.2}$$

$$\begin{aligned} \mathcal{C}_\beta = \beta_1(30'|30'|02'|02'|02') + \beta_2(30'|21'|11'|02'|02') + \\ + \beta_3(30'|20'|12'|02'|02') + \beta_4(30'|20'|11'|03'|02') + \\ + \beta_5(30'|12'|11'|11'|02') + \beta_6(30'|11'|11'|11'|03') + \\ + \beta_7(21'|21'|20'|02'|02') + \beta_8(21'|21'|11'|11'|02') + \\ + \beta_9(21'|20'|20'|03'|02') + \beta_{10}(21'|20'|12'|11'|02') + \\ + \beta_{11}(21'|20'|11'|11'|03') + \beta_{12}(21'|12'|11'|11'|11') + \\ + \beta_{13}(20'|20'|20'|03'|03') + \beta_{14}(20'|20'|12'|12'|02') + \\ + \beta_{15}(20'|20'|12'|11'|03') + \beta_{16}(20'|12'|12'|11'|11') \ , \end{aligned} \tag{3.3}$$

$$\begin{aligned} \mathcal{C}_\gamma = \gamma_1\,(30'|30'|03'|03') + \gamma_2\,(30'|21'|12'|03') + \gamma_3\,(30'|12'|12'|12') + \\ + \gamma_4\,(21'|21'|21'|03') + \gamma_5\,(21'|21'|12'|12') \ , \end{aligned} \tag{3.4}$$

$$\mathcal{C}_\delta = \delta_1\,(20'|20'|02'|02') + \delta_2\,(20'|11'|11'|02') + \delta_3\,(11'|11'|11'|11') \ , \tag{3.5}$$

$$\begin{aligned} \mathcal{C}_\epsilon = \epsilon_1\,(30'|12'|02) + \epsilon_2\,(30'|11'|03) + \epsilon_3\,(21'|21'|02) + \\ + \epsilon_4\,(21'|20'|03) + \epsilon_5\,(21'|12'|11) + \epsilon_6\,(20'|12'|12) \ , \end{aligned} \tag{3.6}$$

$$\mathcal{C}_\zeta = \zeta_1\,(20'|02') + \zeta_2\,(11'|11') \ . \tag{3.7}$$

There are six contributions in the Ansatz; each contribution of the above expression will be called *sector*. A sector can be labeled by T^i_{jk}, meaning the contribution with i-factors in the

Weyl-ordered product and index order sum j, k. With this notation, the six sectors are denoted as

$$\mathcal{C}_\alpha := T^6_{66};\ \mathcal{C}_\beta := T^5_{66};\ \mathcal{C}_\gamma := T^4_{66};\ \mathcal{C}_\delta := T^4_{44};\ \mathcal{C}_\epsilon := T^3_{44};\ \mathcal{C}_\zeta := T^2_{22}\ . \tag{3.8}$$

According to the index sum in each Weyl-ordered product, we can distinguish three groups in (3.1):

* A 6, 6-group formed by the sectors $\mathcal{C}_\alpha$, $\mathcal{C}_\beta$ and $\mathcal{C}_\gamma$. $\mathcal{C}_\alpha$ contains products of level-2 operators, in $\mathcal{C}_\gamma$ only level-3 operators appear, while the sector $\mathcal{C}_\beta$ comprises sixteen 5-products mixing level-2 and -3 operators;

* The second 4, 4-group with sectors $\mathcal{C}_\delta$ and $\mathcal{C}_\epsilon$, appearing in the first of them only level-2 operators, in the second again mixed products with level-2 and -3 operators;

* The 2, 2-sector $\mathcal{C}_\zeta$ alone, containing only level-2 operators, and giving account of the Casimir for the $\mathfrak{sl}(2)$ subalgebra.

As we will see in the coming sections, the written order of the Casimir candidate is related to the dimensions of the coefficients in terms of $\hbar$. Thus, the sectors $\mathcal{C}_\alpha$, $\mathcal{C}_\beta$ and $\mathcal{C}_\gamma$ have coefficients whose order is $\hbar^0$, and they will remain in the classical case when $\hbar \to 0$; however, $\mathcal{C}_\delta$, $\mathcal{C}_\epsilon$ and $\mathcal{C}_\zeta$ will be quantum corrections of order $\hbar^2$ and $\hbar^4$, respectively. Those statements will be fully confirmed by the calculations.

The main task to be accomplished in this section is to determine all the coefficients in the expressions (3.1), or at least as many as possible. How will this be achieved? In short terms, calculating commutators with some basis operators and by imposing its vanishing, a system of equations will be obtained. Due to the number of coefficients, it is expected an over-determined system, which evidently will complicate the algebra, and for this reason a mathematical software will be used, e.g. Maple™.

3.2 Calculations for $[\mathcal{C}, B'_{20}] = 0$

We show the procedure for determining the coefficients in the $\mathcal{C}_\beta$-sector, the same method gilts for all other sectors and cases $[\mathcal{C}, 30'] = 0$ and $[\mathcal{C}, 21'] = 0$; for the sake of simplicity, all the primes will be omitted. Since B_{20} , B_{02} and B_{11} form a closed subgroup (see Table 2), we can choose any of these operators, we will work with B_{20}. Let us recall the full expression for $\mathcal{C}_\beta$:

$$\begin{aligned}\mathcal{C}_\beta = {}& \beta_1(30|30|02|02|02) + \beta_2(30|21|11|02|02) + \beta_3(30|20|12|02|02) + \beta_4(30|20|11|03|02)+ \\ & + \beta_5(30|12|11|11|02) + \beta_6(30|11|11|11|03) + \beta_7(21|21|20|02|02) + \beta_8(21|21|11|11|02)+ \\ & + \beta_9(21|20|20|03|02) + \beta_{10}(21|20|12|11|02) + \beta_{11}(21|20|11|11|03) + \beta_{12}(21|12|11|11|11)+ \\ & + \beta_{13}(20|20|20|03|03) + \beta_{14}(20|20|12|12|02) + \beta_{15}(20|20|12|11|03) + \beta_{16}(20|12|12|11|11)\end{aligned} \quad (3.9)$$

For each term we determine the value of the commutator with $B_{20} := (20)$; for example:

$$[(30|30|02|02|02), 20] = 3(30|30|02|02|[02, 20]) = -12i(30|30|11|02|02)\ . \quad (3.10)$$

$(A\|B\|C\|D\|E) \in \mathcal{C}_\beta$	$i^{-1}\,[(A\|B\|C\|D\|E), 20]$
(30|30|02|02|02)	−12(30|30|11|02|02)
(30|21|11|02|02)	−2(30|30|11|02|02) − 2(30|21|20|02|02) − 8(30|21|11|11|02)
(30|20|12|02|02)	−4(30|21|20|02|02) − 8(30|20|12|11|02)
(30|20|11|03|02)	−2(30|20|20|03|02) − 6(30|20|12|11|02) − 4(30|20|11|11|03)
(30|12|11|11|02)	−4(30|21|11|11|02) − 4(30|20|12|11|02) − 4(30|12|11|11|11)
(30|11|11|11|03)	−6(30|20|11|11|03) − 6(30|12|11|11|11)
(21|21|20|02|02)	−4(30|21|20|02|02) − 8(21|21|20|11|02)
(21|21|11|11|02)	−4(30|21|11|11|02) − 4(21|21|20|11|02) − 4(21|21|11|11|11)
(21|20|20|03|02)	−2(30|20|20|03|02) − 6(21|20|20|12|02) − 4(21|20|20|11|03)
(21|20|12|11|02)	−2(30|20|12|11|02) − 4(21|21|20|11|02) − 2(21|20|20|12|02) − 4(21|20|12|11|11)
(21|20|11|11|03)	−2(30|20|11|11|03) − 4(21|20|20|11|03) − 6(21|20|12|11|11)
(21|12|11|11|11)	−2(30|12|11|11|11) − 4(21|21|11|11|11) − 6(21|20|12|11|11)
(20|20|20|03|03)	−12(20|20|20|12|03)
(20|20|12|12|02)	−8(21|20|20|12|02) − 4(20|20|12|12|11)
(20|20|12|11|03)	−4(21|20|20|11|03) − 2(20|20|20|12|03) − 6(20|20|12|12|11)
(20|12|12|11|11)	−8(21|20|12|11|11) − 4(20|20|12|12|11)

Table 3: Contributions from $[\mathcal{C}_\beta, 20] = 0$.

The final results are shown in Table 3. Collecting similar terms and ordering from the greatest to the lowest Weyl-ordered product (according to their indices), we arrive to the following conditions for the coefficients β_i:

$$(30|30|11|02|02) \quad : \quad 12\beta_1 + 2\beta_2 = 0 \tag{3.11}$$

$$(30|21|20|02|02) \quad : \quad 2\beta_2 + 4\beta_3 + 4\beta_7 = 0 \tag{3.12}$$

$$(30|21|11|11|02) \quad : \quad 8\beta_2 + 4\beta_5 + 4\beta_8 = 0 \tag{3.13}$$

$$(30|20|20|03|02) \quad : \quad 2\beta_4 + 2\beta_9 = 0 \tag{3.14}$$

$$(30|20|12|11|02) \quad : \quad 8\beta_3 + 6\beta_4 + 4\beta_5 + 2\beta_{10} = 0 \tag{3.15}$$

$$(30|20|11|11|03) \quad : \quad 4\beta_4 + 6\beta_6 + 2\beta_{11} = 0 \tag{3.16}$$

$$(30|12|11|11|11) \quad : \quad 4\beta_5 + 6\beta_6 + 2\beta_{12} = 0 \tag{3.17}$$

$$(21|21|20|11|02) \quad : \quad 8\beta_7 + 4\beta_8 + 4\beta_{10} = 0 \tag{3.18}$$

$$(21|21|11|11|11) \quad : \quad 4\beta_8 + 4\beta_{12} = 0 \tag{3.19}$$

$$(21|20|20|12|02) \quad : \quad 6\beta_9 + 2\beta_{10} + 8\beta_{14} = 0 \tag{3.20}$$

$$(21|20|20|11|03) \quad : \quad 4\beta_9 + 4\beta_{11} + 4\beta_{15} = 0 \tag{3.21}$$

$$(21|20|12|11|11) \quad : \quad 4\beta_{10} + 6\beta_{11} + 6\beta_{12} + 8\beta_{16} = 0 \tag{3.22}$$

$$(20|20|20|12|03) \quad : \quad 12\beta_{13} + 2\beta_{15} = 0 \tag{3.23}$$

$$(20|20|12|12|11) \quad : \quad 4\beta_{14} + 6\beta_{15} + 4\beta_{16} = 0 \tag{3.24}$$

After losing the system of equations, the solution can be parametrized in the following form:

$$\beta = \beta_1 \left(1, \; -6, \; 0, \; 0, \; 6, \; -2, \; 3, \; 6, \; 0, \; -12, \; 6, \; -6, \; 1, \; 3, \; -6, \; 6 \right)^T + \beta_2 \left(0, \; 0, \; 1, \; -1, \; -1, \; 1, \; -1, \; 1, \; 1, \; 1, \; -1, \; -1, \; 0, \; -1, \; 0, \; 1 \right)^T . \tag{3.25}$$

The procedure is exactly the same for the sectors $\mathcal{C}_\alpha$, $\mathcal{C}_\gamma$, $\mathcal{C}_\delta$, $\mathcal{C}_\epsilon$ und $\mathcal{C}_\zeta$. The tables 4 to 8 summarize the results after taking commutator with B_{20}.

$(A\|B\|C\|D\|E\|F) \in \mathcal{C}_\alpha$	$i^{-1}\left[(A\|B\|C\|D\|E\|F), 20\right]$
(20|20|20|02|02|02)	-12(20|20|20|11|02|02)
(20|20|11|11|02|02)	-4(20|20|20|11|02|02) $-$ 8(20|20|11|11|11|02)
(20|11|11|11|11|02)	-8(20|20|11|11|11|02) $-$ 4(20|11|11|11|11|11)
(11|11|11|11|11|11)	-12(20|11|11|11|11|11)

Table 4: Contributions from $[\mathcal{C}_\alpha, 20] = 0$.

$(A\|B\|C\|D) \in \mathcal{C}_\gamma$	$i^{-1}\,[(A\|B\|C\|D), 20]$
$(30\|30\|03\|03)$	$-12i(30\|30\|12\|03)$
$(30\|21\|12\|03)$	$-2i(30\|30\|12\|03) - 4i(30\|21\|21\|03) - 6i(30\|21\|12\|12)$
$(30\|12\|12\|12)$	$-12i(30\|21\|12\|12)$
$(21\|21\|21\|03)$	$-6i(30\|21\|21\|03) - 6i(21\|21\|21\|12)$
$(21\|21\|12\|12)$	$-4i(30\|21\|12\|12) - 8i(21\|21\|21\|12)$

Table 5: Contributions from $[\mathcal{C}_\gamma, 20] = 0$.

$(A\|B\|C\|D) \in \mathcal{C}_\delta$	$i^{-1}\,[(A\|B\|C\|D), 20]$
$(20\|20\|02\|02)$	$-8(20\|20\|11\|02)$
$(20\|11\|11\|02)$	$-4(20\|20\|11\|02) - 4(20\|11\|11\|11)$
$(11\|11\|11\|11)$	$-8i(20\|11\|11\|11)$

Table 6: Contributions from $[\mathcal{C}_\delta, 20] = 0$.

$(A\|B\|C) \in \mathcal{C}_\epsilon$	$i^{-1}\,[(A\|B\|C), 20]$
$(30\|12\|02)$	$-4(30\|21\|02) - 4(30\|12\|11)$
$(30\|11\|03)$	$-2(30\|20\|03) - 6(30\|12\|11)$
$(21\|21\|02)$	$-4(30\|21\|02) - 4(21\|21\|11)$
$(21\|20\|03)$	$-2(30\|20\|03) - 6(21\|20\|12)$
$(21\|12\|11)$	$-2(30\|12\|11) - 4(21\|21\|11) - 2(21\|20\|12)$
$(20\|12\|12)$	$-8(21\|20\|12)$

Table 7: Contributions from $[\mathcal{C}_\epsilon, 20] = 0$.

$(A\|B) \in \mathcal{C}_\zeta$	$i^{-1}\,[(A\|B), 20]$
$(20\|02)$	$-4i(20\|11)$
$(11\|11)$	$-4i(20\|11)$

Table 8: Contributions from $[\mathcal{C}_\zeta, 20] = 0$.

From the calculations, the Casimir Operator takes the form

$$
\begin{aligned}
\mathcal{C} = \alpha \Big[\, & 3\,(20|20|11|11|02|02) - (20|20|20|02|02|02) - 3\,(20|11|11|11|11|02) + \\
& + (11|11|11|11|11|11) \; \Big] +
\end{aligned}
$$

$$
+ [\beta_1]\,(30|30|02|02|02) + [-6\beta_1]\,(30|21|11|02|02) + [\beta_2]\,(30|20|12|02|02) + [-\beta_2]\,(30|20|11|03|02) +
$$

$$
+ [6\beta_1 - \beta_2]\,(30|12|11|11|02) + [-2\beta_1 + \beta_2]\,(30|11|11|11|03) + [3\beta_1 - \beta_2]\,(21|21|20|02|02) +
$$

$$
+ [6\beta_1 + \beta_2]\,(21|21|11|11|02) + [\beta_2]\,(21|20|20|03|02) + [-12\beta_1 + \beta_2]\,(21|20|12|11|02) +
$$

$$
+ [6\beta_1 - \beta_2]\,(21|20|11|11|03) + [-6\beta_1 - \beta_2]\,(21|12|11|11|11) + [\beta_1]\,(20|20|20|03|03) +
$$

$$
+ [3\beta_1 - \beta_2]\,(20|20|12|12|02) + [-6\beta_1]\,(20|20|12|11|03) + [6\beta_1 + \beta_2]\,(20|12|12|11|11) +
$$

$$
+ \gamma \left[-\frac{1}{6}\,(30|30|03|03) + (30|21|12|03) - \frac{2}{3}\,(30|12|12|12) - \frac{2}{3}\,(21|21|21|03) + \frac{1}{2}\,(21|21|12|12) \right] +
$$

$$
+ \delta \Big[\; (20|20|02|02) - 2\,(20|11|11|02) + (11|11|11|11) \Big] +
$$

$$
+ \epsilon \Big[\; -(30|12|02) + (30|11|03) + (21|21|02) - (21|20|03) - (21|12|11) + (20|12|12) \; \Big] +
$$

$$
+ \zeta \Big[\; (20|02) - (11|11) \Big] \; . \qquad (3.26)
$$

3.3 Calculations for $[\mathcal{C}, B'_{30}] = 0$

The next step is to fix the remaining constants α, β_1, β_2, γ, δ, ϵ and ζ by imposing $[\mathcal{C}, B'_{30}] = 0$. The procedure is exactly the same as in the case for $[\mathcal{C}, B'_{20}] = 0$, but now additional terms will appear, which increase the complexity when determining the respective system of equations. The tables 9 to 14 summarize the results of the calculations:

$(A\|B\|C\|D\|E\|F) \in \mathcal{C}_\alpha$	$i^{-1}\,[(A\|B\|C\|D\|E\|F), 30]$
$(20\|20\|20\|02\|02\|02)$	$18(21\|20\|20\|20\|02\|02)$
$(20\|20\|11\|11\|02\|02)$	$-18(30\|20\|20\|11\|02\|02) - 36(21\|20\|20\|11\|11\|02)$
$(20\|11\|11\|11\|11\|02)$	$36(30\|20\|11\|11\|11\|02) + 18(21\|20\|11\|11\|11\|11)$
$(11\|11\|11\|11\|11\|11)$	$-18(30\|11\|11\|11\|11\|11)$

Table 9: Contributions from $[\mathcal{C}_\alpha, 30] = 0$.

$(A\|B\|C\|D) \in \mathcal{C}_\gamma$	$i^{-1}\,[(A\|B\|C\|D), 30]$
$-\frac{1}{6}(30\|30\|03\|03)$	$-3i(\frac{1}{6}(20\|02)\|30\|30\|03) + 3i(\frac{1}{3}(11\|11)\|30\|30\|03) + \frac{4i}{3}(30\|30\|03)$
$(30\|21\|12\|03)$	$-3i(\frac{1}{6}(20\|20)\|30\|12\|03) - 6i(\frac{1}{6}(20\|11)\|30\|21\|03) + 9i(\frac{1}{6}(20\|02)\|30\|21\|12)+$ $-9i(\frac{1}{3}(11\|11)\|30\|21\|12) - 4i(30\|21\|12)$
$-\frac{2}{3}(30\|12\|12\|12)$	$12i(\frac{1}{6}(20\|11)\|30\|12\|12)$
$-\frac{2}{3}(21\|21\|21\|03)$	$6i(\frac{1}{6}(20\|20)\|21\|21\|03) - 6i(\frac{1}{6}(20\|02)\|21\|21\|21) + 6i(\frac{1}{3}(11\|11)\|21\|21\|21)+$ $+\frac{8i}{3}(21\|21\|21)$
$\frac{1}{2}(21\|21\|12\|12)$	$-3i(\frac{1}{6}(20\|20)\|21\|12\|12) - 6i(\frac{1}{6}(20\|11)\|21\|21\|12)$

Table 10: Contributions from $[\mathcal{C}_\gamma, 30] = 0$.

$(A\|B\|C\|D) \in \mathcal{C}_\delta$	$i^{-1}\,[(A\|B\|C\|D), 30]$
$(20\|20\|02\|02)$	$-12(21\|20\|20\|02)$
$-2(20\|11\|11\|02)$	$12(30\|20\|11\|02) + 12(21\|20\|11\|11)$
$(11\|11\|11\|11)$	$-12(30\|11\|11\|11)$

Table 11: Contributions from $[\mathcal{C}_\delta, 30] = 0$.

$(A\|B\|C\|D\|E) \in \mathcal{C}_\beta$	$i^{-1}\,[(A\|B\|C\|D\|E), 30]$
$(30\|30\|02\|02\|02)$	$-18(30\|30\|21\|02\|02)$
$(30\|21\|11\|02\|02)$	$-3(\frac{1}{6}(20\|20)\|30\|11\|02\|02) - 3(30\|30\|21\|02\|02) - 12(30\|21\|21\|11\|02)$
$(30\|20\|12\|02\|02)$	$-6(\frac{1}{6}(20\|11)\|30\|20\|02\|02) - 12(30\|21\|20\|12\|02)$
$(30\|20\|11\|03\|02)$	$+9(\frac{1}{6}(20\|02)\|30\|20\|11\|02) - 9(\frac{1}{3}(11\|11)\|30\|20\|11\|02)+$ $-3(30\|30\|20\|03\|02) - 6(30\|21\|20\|11\|03)+$ $-4(30\|20\|11\|02)$
$(30\|12\|11\|11\|02)$	$-6(\frac{1}{6}(20\|11)\|30\|11\|11\|02) - 6(30\|30\|12\|11\|02) - 6(30\|21\|12\|11\|11)$
$(30\|11\|11\|11\|03)$	$+9(\frac{1}{6}(20\|02)\|30\|11\|11\|11) - 9(\frac{1}{3}(11\|11)\|30\|11\|11\|11)+$ $-9(30\|30\|11\|11\|03) - 4(30\|11\|11\|11)$
$(21\|21\|20\|02\|02)$	$-6(\frac{1}{6}(20\|20)\|21\|20\|02\|02) - 12(21\|21\|21\|20\|02)$
$(21\|21\|11\|11\|02)$	$-6(\frac{1}{6}(20\|20)\|21\|11\|11\|02) - 6(30\|21\|21\|11\|02) - 6(21\|21\|21\|11\|11)$
$(21\|20\|20\|03\|02)$	$-3(\frac{1}{6}(20\|20)\|20\|20\|03\|02) + 9(\frac{1}{6}(20\|02)\|21\|20\|20\|02)+$ $-9(\frac{1}{3}(11\|11)\|21\|20\|20\|02) - 6(21\|21\|20\|20\|03)+$ $-4(21\|20\|20\|02)$
$(21\|20\|12\|11\|02)$	$-3(\frac{1}{6}(20\|20)\|20\|12\|11\|02) - 6(\frac{1}{6}(20\|11)\|21\|20\|11\|02)+$ $-3(30\|21\|20\|12\|02) - 6(21\|21\|20\|12\|11)$
$(21\|20\|11\|11\|03)$	$-3(\frac{1}{6}(20\|20)\|20\|11\|11\|03) + 9(\frac{1}{6}(20\|02)\|21\|20\|11\|11)+$ $-9(\frac{1}{3}(11\|11)\|21\|20\|11\|11) - 6(30\|21\|20\|11\|03)+$ $-4(21\|20\|11\|11)$
$(21\|12\|11\|11\|11)$	$-3(\frac{1}{6}(20\|20)\|12\|11\|11\|11) - 6(\frac{1}{6}(20\|11)\|21\|11\|11\|11) - 9(30\|21\|12\|11\|11)$
$(20\|20\|20\|03\|03)$	$+18(\frac{1}{6}(20\|02)\|20\|20\|20\|03) - 18(\frac{1}{3}(11\|11)\|20\|20\|20\|03) - 8(20\|20\|20\|03)$
$(20\|20\|12\|12\|02)$	$-12(\frac{1}{6}(20\|11)\|20\|20\|12\|02) - 6(21\|20\|20\|12\|12)$
$(20\|20\|12\|11\|03)$	$-6(\frac{1}{6}(20\|11)\|20\|20\|11\|03) + 9(\frac{1}{6}(20\|02)\|20\|20\|12\|11)+$ $-9(\frac{1}{3}(11\|11)\|20\|20\|12\|11) - 3(30\|20\|20\|12\|03)+$ $-4(20\|20\|12\|11)$
$(20\|12\|12\|11\|11)$	$-12(\frac{1}{6}(20\|11)\|20\|12\|11\|11) - 6(30\|20\|12\|12\|11)$

Table 12: Contributions from $[\mathcal{C}_\beta, 30] = 0$.

$(A\|B\|C) \in \mathcal{C}_\epsilon$	$i^{-1}\,[(A\|B\|C), 30]$
(30|12|02)	$6(\frac{1}{6}(20\|11)\|30\|02) + 6(30\|21\|12)$
(30|11|03)	$9(\frac{1}{6}(20\|02)\|30\|11) - 9(\frac{1}{3}(11\|11)\|30\|11) - 3(30\|30\|03) - 4(30\|11)$
(21|21|02)	$-6(\frac{1}{6}(20\|20)\|21\|02) - 6(21\|21\|21)$
(21|20|03)	$3(\frac{1}{6}(20\|20)\|20\|03) - 9(\frac{1}{6}(20\|02)\|21\|20) + 9(\frac{1}{3}(11\|11)\|21\|20) + 4(21\|20)$
(21|12|11)	$3(\frac{1}{6}(20\|20)\|12\|11) + 6(\frac{1}{6}(20\|11)\|21\|11) + 3(30\|21\|12)$
(20|12|12)	$-12(\frac{1}{6}(20\|11)\|20\|12)$

Table 13: Contributions from $[\mathcal{C}_\epsilon, 30] = 0$.

$(A\|B) \in \mathcal{C}_\zeta$	$i^{-1}\,[(A\|B), 30]$
(20|02)	$-6(21\|20)$
$-(11\|11)$	$6(30\|11)$

Table 14: Contributions from $[\mathcal{C}_\zeta, 30] = 0$.

3.3.1 Calculations for $[\mathcal{C}, B'_{21}] = 0$

$(A\|B\|C\|D\|E\|F) \in \mathcal{C}_\alpha$	$i^{-1}\,[(A\|B\|C\|D\|E\|F), 21]$
(20|20|20|02|02|02)	$-6(30\|20\|20\|02\|02\|02) + 12(20\|20\|20\|12\|02\|02)$
(20|20|11|11|02|02)	$12(30\|20\|11\|11\|02\|02) - 6(21\|20\|20\|11\|02\|02) - 24(20\|20\|12\|11\|11\|02)$
(20|11|11|11|11|02)	$-6(30\|11\|11\|11\|11\|02) + 12(21\|20\|11\|11\|11\|02) + 12(20\|12\|11\|11\|11\|11)$
(11|11|11|11|11|11)	$-6(21\|11\|11\|11\|11\|11)$

Table 15: Contributions from $[\mathcal{C}_\alpha, 21] = 0$.

$(A\|B\|C\|D\|E) \in \mathcal{C}_\beta$	$i^{-1}\,[(A\|B\|C\|D\|E), 21]$
(30|30|02|02|02)	$6(\frac{1}{6}(20\|20)\|30\|02\|02\|02) - 12(30\|30\|12\|02\|02)$
(30|21|11|02|02)	$3(\frac{1}{6}(20\|20)\|21\|11\|02\|02) - (30\|21\|21\|02\|02) - 8(30\|21\|12\|11\|02)$
(30|20|12|02|02)	$3(\frac{1}{6}(20\|20)\|20\|12\|02\|02) - (\frac{5}{6}(20\|02)\|30\|20\|02\|02) + (\frac{1}{3}(11\|11)\|30\|20\|02\|02)+$ $+2(30\|30\|12\|02\|02) - 8(30\|20\|12\|12\|02) - \frac{4}{3}(30\|20\|02\|02)$
(30|20|11|03|02)	$3(\frac{1}{6}(20\|20)\|20\|11\|03\|02) - 6(\frac{1}{6}(11\|02)\|30\|20\|11\|02) + 2(30\|30\|11\|03\|02)+$ $-(30\|21\|20\|03\|02) - 4(30\|20\|12\|11\|03)$
(30|12|11|11|02)	$3(\frac{1}{6}(20\|20)\|12\|11\|11\|02) - (\frac{5}{6}(20\|02)\|30\|11\|11\|02) + (\frac{1}{3}(11\|11)\|30\|11\|11\|02)+$ $-2(30\|21\|12\|11\|02) - 4(30\|12\|12\|11\|11) - \frac{4}{3}(30\|11\|11\|02)$
(30|11|11|11|03)	$3(\frac{1}{6}(20\|20)\|11\|11\|11\|03) - 6(\frac{1}{6}(11\|02)\|30\|11\|11\|11) - 3(30\|21\|11\|11\|03)$
(21|21|20|02|02)	$2(30\|21\|21\|02\|02) - 8(21\|21\|20\|12\|02)$
(21|21|11|11|02)	$-2(21\|21\|21\|11\|02) - 4(21\|21\|12\|11\|11)$
(21|20|20|03|02)	$-6(\frac{1}{6}(11\|02)\|21\|20\|20\|02) + 4(30\|21\|20\|03\|02) - 4(21\|20\|20\|12\|03)$
(21|20|12|11|02)	$-(\frac{5}{6}(20\|02)\|21\|20\|11\|02) + (\frac{1}{3}(11\|11)\|21\|20\|11\|02) + 2(30\|21\|12\|11\|02)+$ $-(21\|21\|20\|12\|02) - 4(21\|20\|12\|12\|11) - \frac{4}{3}(21\|20\|11\|02)$
(21|20|11|11|03)	$-6(\frac{1}{6}(11\|02)\|21\|20\|11\|11) + 2(30\|21\|11\|11\|03) - 2(21\|21\|20\|11\|03)$
(21|12|11|11|11)	$-(\frac{5}{6}(20\|02)\|21\|11\|11\|11) + (\frac{1}{3}(11\|11)\|21\|11\|11\|11) - 3(21\|21\|12\|11\|11)+$ $-\frac{4}{3}(21\|11\|11\|11)$
(20|20|20|03|03)	$-12(\frac{1}{6}(11\|02)\|20\|20\|20\|03) + 6(30\|20\|20\|03\|03)$
(20|20|12|12|02)	$-2(\frac{5}{6}(20\|02)\|20\|20\|12\|02) + 2(\frac{1}{3}(11\|11)\|20\|20\|12\|02) + 4(30\|20\|12\|12\|02)+$ $-4(20\|20\|12\|12\|12) - \frac{8}{3}(20\|20\|12\|02)$
(20|20|12|11|03)	$-(\frac{5}{6}(20\|02)\|20\|20\|11\|03) + (\frac{1}{3}(11\|11)\|20\|20\|11\|03) - 6(\frac{1}{6}(11\|02)\|20\|20\|12\|11)+$ $+4(30\|20\|12\|11\|03) - (21\|20\|20\|12\|03) - \frac{4}{3}(20\|20\|11\|03)$
(20|12|12|11|11)	$-2(\frac{5}{6}(20\|02)\|20\|12\|11\|11) + 2(\frac{1}{3}(11\|11)\|20\|12\|11\|11) + 2(30\|12\|12\|11\|11)+$ $-2(21\|20\|12\|12\|11) - \frac{8}{3}(20\|12\|11\|11)$

Table 16: Contributions from $[\mathcal{C}_\beta, 21] = 0$.

$(A\|B\|C\|D) \in \mathcal{C}_\gamma$	$i^{-1}\,[(A\|B\|C\|D), 21]$
$-\frac{1}{6}(30\|30\|03\|03)$	$-(\frac{1}{6}(20\|20)\|30\|03\|03) + 2(\frac{1}{6}(11\|02)\|30\|30\|03)$
$(30\|21\|12\|03)$	$3(\frac{1}{6}(20\|20)\|21\|12\|03) - (\frac{5}{6}(20\|02)\|30\|21\|03) + (\frac{1}{3}(11\|11)\|30\|21\|03)+$ $-6(\frac{1}{6}(11\|02)\|30\|21\|12) - \frac{4}{3}(30\|21\|03)$
$-\frac{2}{3}(30\|12\|12\|12)$	$-2(\frac{1}{6}(20\|20)\|12\|12\|12) + 2(\frac{5}{6}(20\|02)\|30\|12\|12) - 2(\frac{1}{3}(11\|11)\|30\|12\|12)+$ $+\frac{8}{3}(30\|12\|12)$
$-\frac{2}{3}(21\|21\|21\|03)$	$4(\frac{1}{6}(11\|02)\|21\|21\|21)$
$\frac{1}{2}(21\|21\|12\|12)$	$-(\frac{5}{6}(20\|02)\|21\|21\|12) + (\frac{1}{3}(11\|11)\|21\|21\|12) - \frac{4}{3}(21\|21\|12)$

Table 17: Contributions from $[\mathcal{C}_\gamma, 21] = 0$.

$(A\|B\|C\|D) \in \mathcal{C}_\delta$	$i^{-1}\,[(A\|B\|C\|D), 21]$
$(20\|20\|02\|02)$	$4(30\|20\|02\|02) - 8(20\|20\|12\|02)$
$-2(20\|11\|11\|02)$	$-4(30\|11\|11\|02) + 4(21\|20\|11\|02) + 8(20\|12\|11\|11)$
$(11\|11\|11\|11)$	$-4(21\|11\|11\|11)$

Table 18: Contributions from $[\mathcal{C}_\delta, 21] = 0$.

$(A\|B\|C) \in \mathcal{C}_\epsilon$	$i^{-1}\,[(A\|B\|C), 21]$
$(30\|12\|02)$	$-(\frac{1}{6}(20\|20)\|12\|02) + (\frac{5}{6}(20\|02)\|30\|02) - (\frac{1}{3}(11\|11)\|30\|02)+$ $+4(30\|12\|12) + \frac{4}{3}(30\|02)$
$(30\|11\|03)$	$(\frac{1}{6}(20\|20)\|11\|03) - 2(\frac{1}{6}(11\|02)\|30\|11) - (30\|21\|03)$
$(21\|21\|02)$	$-4(21\|21\|12)$
$(21\|20\|03)$	$+2(\frac{1}{6}(11\|02)\|21\|20) - 2(30\|21\|03)$
$(21\|12\|11)$	$(\frac{5}{6}(20\|02)\|21\|11) - (\frac{1}{3}(11\|11)\|21\|11) + (21\|21\|12)+$ $+\frac{4}{3}(21\|11)$
$(20\|12\|12)$	$-2(\frac{5}{6}(20\|02)\|20\|12) + 2(\frac{1}{3}(11\|11)\|20\|12)2(30\|12\|12)+$ $-\frac{8}{3}(20\|12)$

Table 19: Contributions from $[\mathcal{C}_\epsilon, 21] = 0$.

$(A\|B) \in \mathcal{C}_\zeta$	$i^{-1}\left[(A\|B), 21\right]$
$(20\|02)$	$2(30\|02) - 4(20\|12)$
$-(11\|11)$	$2(21\|11)$

Table 20: Contributions from $[\mathcal{C}_\zeta, 21] = 0$.

From the results in the tables 9 to 20 it can be observed the presence of a new type of Weyl-ordered product, the so-called composed product, because of the level-3 commutators, which generate a constant term plus a Weyl-ordered product of level-2 operators, e.g.

$$(30'|22'|11') = -\frac{1}{6}(30'|(20'|02')|11') + \frac{1}{3}(30'|(11'|11')|11') - \frac{13'}{9}(30'|11') \ . \tag{3.27}$$

What can be done to handle such products?

* In the work of Isakov and Leinaas [14], they also study a Calogero model with interaction parameter λ, and starting from the commutator relations (K_{ij} corresponds to our s_{ij} 2-particle permutation operator)

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0 \ , \tag{3.28}$$

$$[a_i, a_j^\dagger] = \delta_{ij}\left(1 + \lambda \sum_i K_{il}\right) - \lambda K_{ij} \ , \tag{3.29}$$

$$K_{ij}K_{jl} = K_{jl}K_{il} = K_{il}K_{ij}, i \neq j, i \neq l, j \neq l \ , \tag{3.30}$$

$$K_{ij}K_{mn} = K_{mn}K_{ij} \forall i \neq j \neq m \neq n \ , \tag{3.31}$$

$$K_{ij}a_j = a_i K_{ij}, \ \ K_{ij}a_j^\dagger = a_i^\dagger K_{ij} \ , \tag{3.32}$$

they define the operators

$$L_{0n} = \sum_i a_i^n \ , \tag{3.33}$$

$$L_{n0} = \sum_i a_i^{\dagger n} \ , \tag{3.34}$$

and obtain results like our commutator formulae, e.g.

$$[L_{m1}, L_{n1}] = (n-m)L_{m+n-1,1} \ , \tag{3.35}$$

but they don't reduce the results from higher level operators to lower level operators, like this example shows

$$L_{31} = \frac{1}{2}\left(a^{\dagger 3}a + a a^{\dagger 3}\right) \ . \tag{3.36}$$

So, applying the definition of the operators B_{kl} in terms of the position and momenta coordinates, the composed products could be expanded, and the resulting expressions should be used for determining a system of equations for the coefficients of the Casimir;

* The second option, which will be used in the next calculations, consists in the reduction of the composed products into some combination of pure Weyl-ordered products of higher or lower order. For this purpose, reduction formulae involving double commutators must be applied, being the general idea behind their derivation explained in the next section.

3.4 Composed Weyl-Products

We proceed to analyze the problem of defining composed (not pure) Weyl-ordered products in terms of pure ones. Let us start by considering the simplest composed product, namely, $(a|(b|c))$; clearly, there is a difference with respect to $(a|b|c)$:

$$\begin{aligned} (a|(b|c)) &= \frac{1}{2}a(b|c) + \frac{1}{2}(b|c)a \\ &= \frac{1}{4}\left[abc + acb + bca + cba\right] \ , \end{aligned} \tag{3.37}$$

$$(a|b|c) = \frac{1}{6}\left[abc + acb + bac + bca + cab + cba\right] \ . \tag{3.38}$$

The problem now is how to relate both products; in the present work this was achieved by means of pure algebraic manipulations, and the detailed proof can be found in the appendix A.6. We only list a few formulae appearing in the calculations from $[\mathcal{C}, 30] = 0$ and $[\mathcal{C}, 21] = 0$:

$$R^a_{bc} = \frac{1}{12}\left([[a,b],c] + [[a,c],b]\right) \ , \tag{3.39}$$

$$(a|(b|c)) = (a|b|c) + R^a_{bc} \ , \tag{3.40}$$

$$(a|b|(c|d)) = (a|b|c|d) + \left[\begin{array}{c} (a|R^b_{cd}) \\ (b|R^a_{cd}) \end{array}\right] + \left(\frac{1}{12}\right)\left[\begin{array}{c} ([a,c]|[b,d]) \\ ([a,d]|[b,c]) \end{array}\right] \ , \tag{3.41}$$

where it is defined

$$\left[\begin{array}{c} (a|R^b_{cd}) \\ (b|R^a_{cd}) \end{array}\right] := (a|R^b_{cd}) + (b|R^a_{cd}) \ , \tag{3.42}$$

and not a matrix;

$$(a|b|c|(d|e)) = (a|b|c|d|e) + \begin{bmatrix} (a|b|R^c_{de}) \\ (a|c|R^b_{de}) \\ (b|c|R^a_{de}) \end{bmatrix} + \left(\frac{1}{12}\right) \begin{bmatrix} (a|[b,d]|[c,e]) + (a|[b,e]|[c,d]) \\ (b|[a,d]|[c,e]) + (b|[a,e]|[c,d]) \\ (c|[a,d]|[b,e]) + (c|[a,e]|[b,d]) \end{bmatrix}$$

$$+ \left(\frac{1}{30}\right) \begin{bmatrix} R^a_{[b,d][c,e]} + R^a_{[b,e][c,d]} \\ R^b_{[a,d][c,e]} + R^b_{[a,e][c,d]} \\ R^c_{[a,d][b,e]} + R^c_{[a,e][b,d]} \end{bmatrix} - \left(\frac{1}{5}\right) \begin{bmatrix} R^{R^a_{de}}_{bc} \\ R^{R^b_{de}}_{ac} \\ R^{R^c_{de}}_{ab} \end{bmatrix} . \quad (3.43)$$

Considering the last formulae, it is clear the complexity when calculating the respective contributions, because of the terms containing double commutators, rests R^a_{bc} and rests from rests like $R^{R^a_{bc}}_{de}$. However, the general principle is easy to understand: for a given composed Weyl-ordered product, e.g. $(a|b|c|d|e|(f|g))$, there will be a first term, the so-called "leading order approximation" $(a|b|c|d|e|f|g)$, terms containing double commutators like $(a|b|c|[d,f]|[e,g])$ and permutations, terms with rests $(a|b|c|d|R^e_{fg})$ plus permutations, etc. To automatize the calculations a simple Maple™program was written, being the results for each composed product summarized in the tables 23 to 30 (see Section A.7 in the appendix). Here we just indicate how the different contributions are related between them by means of diagrams. Black thick arrows will be used for "normal" or "pure" Weyl-Products like (30|20|12) and leading order approximations e.g. $(30|(20|20)|03) \sim (30|20|20|03)$, while color thick arrows will indicate the contributions from "composed" or "dirty" Weyl-Products using the reduction formulae of the appendix A.6. The figures 1 and 2 describe the new scenario, for both cases $[\mathcal{C}, 30'] = 0$ and $[\mathcal{C}, 21'] = 0$. From both figures it is clear, there will be equations connecting different sectors:

* Pure products in $\mathcal{C}_\alpha$ and composed terms from $\mathcal{C}_\beta$;
* Pure products in $\mathcal{C}_\beta$ and composed terms from $\mathcal{C}_\gamma$;
* Contributions relating $\mathcal{C}_\beta$, $\mathcal{C}_\delta$ and $\mathcal{C}_\epsilon$;
* Pure products in $\mathcal{C}_\alpha$, and contributions of reduced terms from $\mathcal{C}_\beta$, $\mathcal{C}_\gamma$ and $\mathcal{C}_\epsilon$.

What about the dimensions of the coefficients? For example, considering the product $(a|(b|c))$, since the rest R^a_{bc} contains a double commutator, this Weyl-ordered product contains terms up to order $\hbar^2$:

$$(a|(b|c)) = (a|b|c) + \hbar^2 R^a_{bc} . \quad (3.44)$$

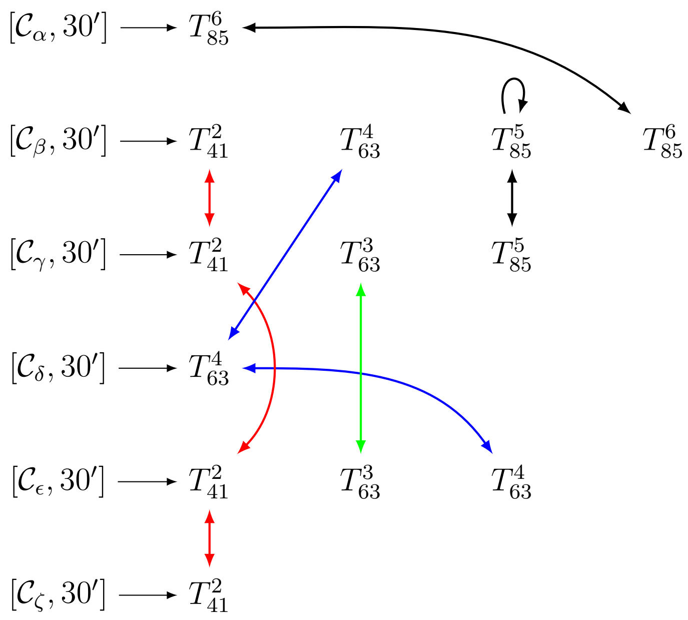

Figure 1: Mixing diagram for $[\mathcal{C}, 30'] = 0$.

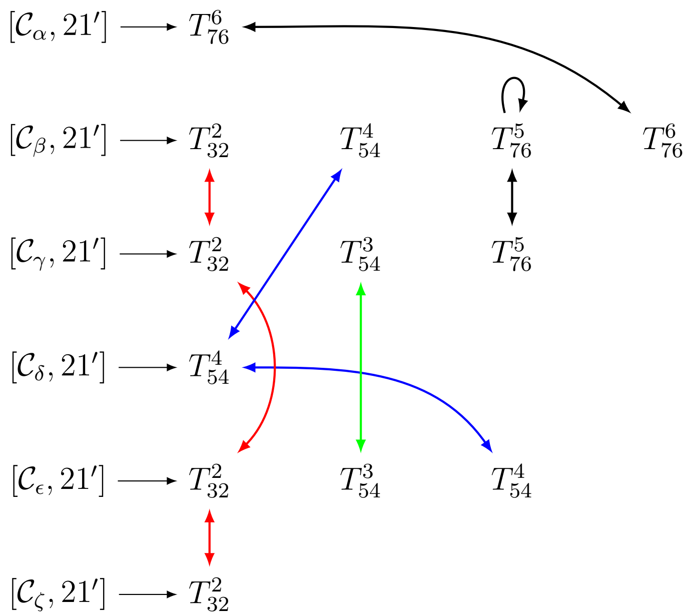

Figure 2: Mixing diagram for $[\mathcal{C}, 21'] = 0$.

Let us analyze one term from $[\mathcal{C}, 30'] = 0$:

$$\begin{aligned}
[(30'|12'|02'), 30'] &= -6i\hbar(31'|30'|02') - 6i\hbar(30'|21'|12') \\
&= -i\hbar(30'|20'|11'|02') - 6i\hbar(30'|21'|12') - \frac{i\hbar^3}{3}(30'|11')\ . \qquad (3.45)
\end{aligned}$$

The same analysis shows

$$(a|b|(c|d)) \sim 1 + \hbar^2\ , \qquad (3.46)$$

$$(a|b|c|(d|e)) \sim 1 + \hbar^2 + \hbar^4\ , \qquad (3.47)$$

$$(a|b|c|d|(e|f)) \sim 1 + \hbar^2 + \hbar^4 \ . \tag{3.48}$$

Then, we confirm what was said in the beginning of this section: The coefficients α, β, γ are of the same order $\hbar^0$, and in the limit $\hbar \to 0$, i.e. the classical case, the contributions of those sectors will remain, not being the case for δ and ϵ, which are of order $\hbar^2$, and for ζ order $\hbar^4$.

After collecting similar terms, applying reduction formulae, reordering, etc. two systems of equations for the coefficients were determined. For example, in the case $[\mathcal{C}, 30'] = 0$, the following matrices give account of the equations obtained; the rows correspond to the different contributions for each Weyl-ordered product (ordering from the greatest to the lowest term, as seen in Section 3.2), and the columns from left to right denote the coefficients α, β_1, β_2, γ, δ, ϵ and ζ, in that order:

* Pure and leading order approximations between α and β_1, β_2:

$$\mathbf{A}_{30'} = \begin{pmatrix} -18 & -\frac{5}{2} & 3 & 0 & 0 & 0 & 0 \\ 36 & \frac{11}{2} & -9 & 0 & 0 & 0 & 0 \\ -18 & -3 & 6 & 0 & 0 & 0 & 0 \\ 18 & \frac{5}{2} & -3 & 0 & 0 & 0 & 0 \\ -36 & -\frac{13}{2} & 15 & 0 & 0 & 0 & 0 \\ 18 & 4 & -12 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 3 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & -9 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -3 & 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{2} & 9 & 0 & 0 & 0 & 0 \\ & & \vdots & & & & \end{pmatrix} . \tag{3.49}$$

* Mixing between β_1, β_2, δ, ϵ,

$$\begin{pmatrix} & & & \vdots & & & \\ 0 & 15 & 6 & 0 & -12 & -\frac{5}{2} & 0 \\ 0 & -15 & -6 & 0 & 12 & \frac{5}{2} & 0 \\ 0 & -14 & 96 & 0 & 12 & 4 & 0 \\ 0 & \frac{44}{3} & -28 & 0 & -12 & -3 & 0 \\ 0 & \frac{1}{3} & 34 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & -1 & -102 & 0 & 0 & -\frac{3}{2} & 0 \\ & & & \vdots & & & \end{pmatrix} , \tag{3.50}$$

* Mixing between β_1, β_2, δ,

$$\begin{pmatrix} & & & \vdots & & & \\ 0 & 3 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -9 & 18 & 1 & 0 & 0 & 0 \\ 0 & 0 & 18 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 12 & -36 & -1 & 0 & 0 & 0 \\ 0 & -15 & 36 & \frac{3}{2} & 0 & 0 & 0 \\ 0 & 15 & 18 & -3 & 0 & 0 & 0 \\ 0 & -6 & -36 & 2 & 0 & 0 & 0 \\ 0 & -6 & 0 & 1 & 0 & 0 & 0 \\ 0 & 12 & -36 & -1 & 0 & 0 & 0 \\ 0 & -6 & -36 & 2 & 0 & 0 & 0 \\ 0 & 6 & -18 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -6 & 72 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -6 & 36 & 0 & 0 & 0 & 0 \\ 0 & 6 & -36 & 0 & 0 & 0 & 0 \\ & & & \vdots & & & \end{pmatrix} , \tag{3.51}$$

* Mixing between the sectors β_1, β_2, δ, ϵ and ζ:

$$\begin{pmatrix} & & & \vdots & & & \\ 0 & -\frac{827}{45} & -\frac{904}{15} & -\frac{112}{15} & 0 & -\frac{38}{3} & -6 \\ 0 & 0 & 0 & 18 & 0 & 9 & 0 \end{pmatrix} , \tag{3.52}$$

A similar matrix can be written for $[\mathcal{C}, 21'] = 0$; after solving the systems of equations with Maple$^{\mathrm{TM}}$, we arrive to one of the central results, namely, the values for the constants defining the free Casimir operator with $N = 3$ particles:

$$\alpha = \alpha,\ \beta_1 = -\frac{3}{2}\alpha;\ \beta_2 = -9\alpha;\ \gamma = -54\alpha;\ \delta = -\frac{69}{2}\alpha;\ \epsilon = 108\alpha;\ \zeta = -\frac{709}{6}\alpha\ . \tag{3.53}$$

We can also normalize this result choosing $\alpha = 6$:

$$\alpha = 6,\ \beta_1 = -9;\ \beta_2 = -54;\ \gamma = -324;\ \delta = -207;\ \epsilon = 648;\ \zeta = -709\ . \tag{3.54}$$

3.5 Classical Limit - Poisson Brackets

A simple test for verifying the last result (3.53) can be done analyzing the classical limit. According to standard texts on classical and quantum mechanics, the relation between commutator and Poisson-bracket is

$$\{A, B\} = \sum_{i=1}^{N} \frac{\partial A}{\partial x_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial x_i} \ , \tag{3.55}$$

$$\frac{i^{-1}}{\hbar}\,[A, B] \to \{A, B\} \ . \tag{3.56}$$

In the classical case, since $\hbar$ tends to zero, all operators commute between themselves, and all terms of order $\hbar$ or higher disappear. Then, the old commutators in Table 2 reduce to the following one:

$\{B_{kl}, B_{mn}\}$	B'_{20}	B'_{02}	B'_{11}	B'_{30}	B'_{21}	B'_{12}	B'_{03}
B'_{20}	0	$4B'_{11}$	$2B'_{20}$	0	$2B'_{30}$	$4B'_{21}$	$6B'_{12}$
B'_{02}	$-4B'_{11}$	0	$-2B'_{02}$	$-6B'_{21}$	$-4B'_{12}$	$-2B'_{03}$	0
B'_{11}	$-2B'_{20}$	$2B'_{02}$	0	$-3B'_{30}$	$-B'_{21}$	B'_{12}	$3B'_{03}$
B'_{30}	0	$6B'_{21}$	$3B'_{30}$	0	$3B'_{40}$	$6B'_{31}$	$9B'_{22}$
B'_{21}	$-2B'_{30}$	$4B'_{12}$	B'_{21}	$-3B'_{40}$	0	$3B'_{22}$	$6B'_{13}$
B'_{12}	$-4B_{21}$	$2B'_{03}$	$-B'_{12}$	$-6B'_{31}$	$-3B'_{22}$	0	$3B'_{04}$
B'_{03}	$-6B'_{12}$	0	$-3B'_{03}$	$-9B'_{22}$	$-6B'_{13}$	$-3B'_{04}$	0

Table 21: Poisson brackets between operators.

Since there are no terms of order $\hbar$ or higher, the mixing between different sectors does not occur. The new behavior of the system can be described as follows:

* There will be equations relating terms from α and β_1, β_2;

* A second set will combine β_1 and β_2 and γ;

* A third set of equations in the same sector;

* Equations associated to δ, ϵ, ζ are all of the form

$$\delta \cdot x = 0, \ x \in \mathbb{R} \ , \tag{3.57}$$

$$\epsilon \cdot y = 0, \ y \in \mathbb{R} \ , \tag{3.58}$$

$$\zeta \cdot z = 0, \ z \in \mathbb{R} \ , \tag{3.59}$$

whose solutions are

$$\delta = \epsilon = \zeta = 0 \ . \tag{3.60}$$

Proceeding exactly like in the general case, i.e. calculating $\{\mathcal{C}, 30'\} = 0$, expanding, collecting terms, etc., the results after solving the respective system of equations show us, that only the sectors $\mathcal{C}_\alpha$, $\mathcal{C}_\beta$, $\mathcal{C}_\gamma$ contribute to the classical Casimir Operator:

$$\alpha = \alpha \ , \beta_1 = -\frac{3}{2}\alpha; \ \ \beta_2 = -9\alpha; \ \ \gamma = -54\alpha; \ \ \delta = 0; \ \ \epsilon = 0; \ \ \zeta = 0 \ , \tag{3.61}$$

again scaling to $\alpha = 6$:

$$\alpha = 6 \ , \beta_1 = -9; \ \ \beta_2 = -54; \ \ \gamma = -324; \ \ \delta = 0; \ \ \epsilon = 0; \ \ \zeta = 0 \ . \tag{3.62}$$

For the last result we can state: In the quantum case, the ground commutators can have contributions of order $\hbar$ or $\hbar^2$, due to this fact the commutator with the Casimir operator exhibits terms of order $\hbar^3$ (sectors β and γ) and $\hbar^5$ (sector α). In the classical case, the commutator is replaced by the Poisson bracket, all operators commute between themselves, and there are no contributions beyond order $\hbar$ in the ground brackets. Then, the only surviving contributions to the classical Casimir operator will come from the sectors α, β and γ, and all the other terms will vanish, which is consistent with the algebraic result (3.61).

Thus, having found a full Casimir operator in its free version for both classical and quantum case, with COM and total momentum decoupled,

$$\mathcal{C} = \alpha \Big[3\,(20'|20'|11'|11'|02'|02') - (20'|20'|20'|02'|02'|02') - 3\,(20'|11'|11'|11'|11'|02') + \\ + (11'|11'|11'|11'|11'|11') \Big] +$$

$$+ \beta \Big[(30'|30'|02'|02'|02') - 6(30'|21'|11'|02'|02') + 6(30'|20'|12'|02'|02') - 6(30'|20'|11'|03'|02') +$$

$$+ 4(30'|11'|11'|11'|03') - 3(21'|21'|20'|02'|02') + 12'(21'|21'|11'|11'|02') + 6(21'|20'|20'|03'|02') +$$

$$- 6(21'|20'|12'|11'|02') - 12'(21'|12'|11'|11'|11') + (20'|20'|20'|03'|03') - 3(20'|20'|12'|12'|02') +$$

$$- 6(20'|20'|12'|11'|03') + 12'(20'|12'|12'|11'|11') \Big] +$$

$$+ \gamma \Big[-\frac{1}{6}\,(30'|30'|03'|03') + (30'|21'|12'|03') - \frac{2}{3}\,(30'|12'|12'|12') - \frac{2}{3}\,(21'|21'|21'|03') + \\ + \frac{1}{2}\,(21'|21'|12'|12') \Big] +$$

$$+ \delta \Big[(20'|20'|02'|02') - 2\,(20'|11'|11'|02') + (11'|11'|11'|11') \Big] +$$

$$+ \epsilon \Big[-(30'|12'|02') + (30'|11'|03') + (21'|21'|02') - (21'|20'|03') - (21'|12'|11') + \\ + (20'|12'|12') \Big] + \\ + \zeta \Big[(20'|02') - (11'|11') \Big] \,, \quad (3.63)$$

being the coefficients (normalized to $\alpha = 6$ and $\beta_2 = 6\beta_1 =: 6\beta$)

$$\alpha = 6;\ \beta_1 = -9;\ \beta_2 = -54;\ \gamma = -324;\ \delta = -207;\ \epsilon := 648;\ \zeta = -709\ , \quad (3.64)$$

we can turn our attention to the problem with interaction g in the Hamiltonian; this task will be developed in the next section.

3.6 Casimir Operator - Interaction Case

Up to now the interaction was absent in the required equations for getting the free Casimir operator. In the last section we have found an operator, and the next logical step is to switch on the interaction g in our problem, that is to say, the full Hamiltonian has to be considered [1, 2]:

$$H = \frac{1}{2}\sum_i p_i^2 + \sum_{i<j} \frac{g(g-1)}{(x^i - x^j)^2} \; . \tag{3.65}$$

The main change concerns to the definition of the B_{kl}-operators:

$$p_k \to \pi_k; \; B_{kl} \to \tilde{B}_{kl} = res\left(\frac{1}{2}\sum_{\mu=1}^{3}\left[x_\mu^k \pi_\mu^l + \pi_\mu^l x_\mu^k\right]\right) \; . \tag{3.66}$$

How do we proceed? In a similar way to the previous case, by finding the new commutator relations, then decoupling the center-of-mass and finally solving a system of equations for the modified constants α, β, γ, δ, ϵ_1, ϵ_2 and ζ.

In what follows, the operators will act on completely symmetric functions under exchange of two particles [1]. Taking this into account, the definitions for B_{kl} with interaction are the following ones:

$$\tilde{B}_{10} = B_{10} \; , \tag{3.67}$$

$$\tilde{B}_{01} = B_{01} \; , \tag{3.68}$$

$$\tilde{B}_{20} = B_{20} \; , \tag{3.69}$$

$$\tilde{B}_{02} = B_{02} + 2g(g-1)\sum_{k<l}\frac{1}{x_{kl}^2} \; , \tag{3.70}$$

$$\tilde{B}_{11} = B_{11} \; , \tag{3.71}$$

$$\tilde{B}_{30} = B_{30} \; , \tag{3.72}$$

$$\tilde{B}_{21} = B_{21} \; , \tag{3.73}$$

$$\tilde{B}_{12} = B_{12} + g(g-1)\sum_{k<l}\frac{x_k + x_l}{x_{kl}^2} \; , \tag{3.74}$$

$$\tilde{B}_{03} = B_{03} + 3g(g-1)\sum_{k<l}\frac{p_k + p_l}{x_{kl}^2} \; , \tag{3.75}$$

being $x_{kl} = x_k - x_l$, as usual. The notation applied here reads: B_{kl} is a free operator, while $\tilde{B}_{kl}$ includes the interaction, but $\tilde{B}'_{kl}$ includes interaction and COM was decoupled using the transformations (2.71), (2.72) and (2.73) seen in Section 2.4.

From the last equations it can be seen, only the operators $\tilde{B}_{02}, \tilde{B}_{12}, \tilde{B}_{03}$ are modified. The next step is to calculate the commutators with those modified operators, to see how the free-case algebra is changed. Here we concentrate, as usual, on the most important commutators, namely

$[\tilde{30},\tilde{03}]$ and $[\tilde{21},\tilde{12}]$, because they will produce constant terms (order $\hbar^3$) which contribute to the sector mixing, either as a direct result or included in the expansions for composed Weyl-ordered products:

$$i^{-1}\left[\tilde{30},\tilde{03}\right] = 9(22) + 9 + 9g(g-1)\sum_{a<b}\frac{x_a^2+x_b^2}{x_{ab}^2}\ , \tag{3.76}$$

$$\imath^{-1}[\tilde{21},\tilde{12}] = 3(22) + 6 + g(g-1)\sum_{a<b}\frac{x_a^2+4x_ax_b+x_b^2}{x_{ab}^2}\ . \tag{3.77}$$

After some straightforward but cumbersome calculations, e.g., when expressing B_{22} in terms of the $\tilde{B}_{kl}$ operators and applying the transformation which decouples COM, the final expression for the commutator $[\tilde{30}',\tilde{03}']$ reads

$$\left[\tilde{30}',\tilde{03}'\right] = -\frac{3}{2}\left(\tilde{20}'|\tilde{02}'\right) + 3(\tilde{11}'|\tilde{11}') - 4 + 9g(g-1)\ , \tag{3.78}$$

and similarly for $[\tilde{21}',\tilde{12}']$

$$\left[\tilde{21}',\tilde{12}'\right] = \frac{5}{6}\left(\tilde{20}'|\tilde{02}'\right) - \frac{1}{3}(\tilde{11}'|\tilde{11}') + \frac{4}{3} - 3g(g-1)\ , \tag{3.79}$$

where the detailed steps for deriving (3.78) are indicated in the Section A.8 of the appendix. For the rest of the commutators, the Table 22 summarizes the new results; there we denote by $W := g(g-1)$ the contribution due to the interaction coupling g. Clearly, the only change due to the presence of the interaction has to do with the addition of a constant term proportional to $g(g-1)$ in the commutators (3.78) and (3.80); in all other cases the commutators (also, their structure constants) remain unchanged, no doubt a quite remarkable effect which will simplify later the calculations.

$i^{-1}[B_{kl}, B_{mn}]$	$(\widetilde{20}')$	$(\widetilde{02}')$	$(\widetilde{11}')$	$(\widetilde{30}')$	$(\widetilde{21}')$	$(\widetilde{12}')$	$(\widetilde{03}')$
$(\widetilde{20}')$	0	$4(\widetilde{11}')$	$2(\widetilde{20}')$	0	$2(\widetilde{30}')$	$4(\widetilde{21}')$	$6(\widetilde{12}')$
$(\widetilde{02}')$	$-4(\widetilde{11}')$	0	$-2(\widetilde{02}')$	$-6(\widetilde{21}')$	$-4(\widetilde{12}')$	$-2(\widetilde{03}')$	0
$(\widetilde{11}')$	$-2(\widetilde{20}')$	$2(\widetilde{02}')$	0	$-3(\widetilde{30}')$	$-(21)'$	$(\widetilde{12}')$	$3(\widetilde{03}')$
$(\widetilde{30}')$	0	$6(\widetilde{21}')$	$3(\widetilde{30}')$	0	$\frac{1}{2}(\widetilde{20}'\vert\widetilde{20}')$	$(\widetilde{20}'\vert\widetilde{11}')$	$-\frac{3}{2}(\widetilde{20}'\vert\widetilde{02}')$ $+3(\widetilde{11}'\vert\widetilde{11}')$ $-4+9W$
$(\widetilde{21}')$	$-2(\widetilde{30}')$	$4(\widetilde{12}')$	$(\widetilde{21}')$	$-\frac{1}{2}(\widetilde{20}'\vert\widetilde{20}')$	0	$+\frac{5}{6}(\widetilde{20}'\vert\widetilde{02}')$ $-\frac{1}{3}(\widetilde{11}'\vert\widetilde{11}')$ $+\frac{4}{3}-3W$	$(\widetilde{11}'\vert\widetilde{02}')$
$(\widetilde{12}')$	$-4(\widetilde{21}')$	$2(\widetilde{03}')$	$-(\widetilde{12}')$	$-(\widetilde{20}'\vert\widetilde{11}')$	$-\frac{5}{6}(\widetilde{20}'\vert\widetilde{02}')$ $+\frac{1}{3}(\widetilde{11}'\vert\widetilde{11}')$ $-\frac{4}{3}+3W$	0	$\frac{1}{2}(\widetilde{02}'\vert\widetilde{02}')$
$(\widetilde{03}')$	$-6(\widetilde{12}')$	0	$-3(\widetilde{03}')$	$+\frac{3}{2}(\widetilde{20}'\vert\widetilde{02}')$ $-3(\widetilde{11}'\vert\widetilde{11}')$ $+4-9W$	$-(\widetilde{11}'\vert\widetilde{02}')$	$-\frac{1}{2}(\widetilde{02}'\vert\widetilde{02}')$	0

Table 22: Commutators with interaction and COM-decoupling.

It is necessary to emphasize the philosophy behind those algebraic calculations:

* Pure commutation relations $[\tilde{B}_{kl}, \tilde{B}_{mn}]$ were determined, including COM-terms; the results coincide with the free case, the only difference appearing in the commutators between level-3 operators, where there are terms proportional to the interaction coupling g;

* As a second step, applying the transformation that decouples COM and total momentum operators, the commutators are recalculated. The results only differ from the free case in $[\widetilde{30}', \widetilde{03}']$ and $[\widetilde{21}', \widetilde{12}']$ by adding a constant depending on the interaction coupling in the form $g(g-1)$;

* Consequently, the calculations for $[\tilde{\mathcal{C}}', \widetilde{30}'] = 0$ and $[\tilde{\mathcal{C}}', \widetilde{21}'] = 0$ must be modified, and a new system of equations is obtained for $\alpha, \dots, \zeta$. Since the change affects commutators with terms of order $\hbar^3$ and $\hbar^5$, i.e. the composed Weyl-ordered products, the new system should only change in the equations mixing β_1, β_2, γ, δ, ϵ and ζ.

We have already seen (Section 3.5) that the sectors α, β, γ don't vanish in the classical free case, and quantum effects are associated to the $\delta-$, $\epsilon-$, and $\zeta-$ sectors. Then, it is reasonable to obtain equations for those quantum sectors where the interaction explicitly appears, while the equations for the classical sector α, β and γ don't experience any changes. The following matrix summarize the modifications due to the presence of the interaction g:

$$\mathbf{A}_{\tilde{30},g} \cdot \vec{X} = \vec{0} \ , \tag{3.80}$$

$$\vec{X} = \begin{pmatrix} \alpha, & \beta_1, & \beta_2, & \gamma, & \delta, & \epsilon, & \zeta \end{pmatrix}^T \tag{3.81}$$

$$\mathbf{A}_{\tilde{30},g} = \begin{pmatrix} & & & \vdots & & & \\ 0 & \frac{827}{45} & \frac{904}{15} & \frac{112}{15} & 0 & \frac{38}{3} - 9W & 6 \\ 0 & 0 & 0 & 18 - 9W & 0 & 9 & 0 \\ 0 & 15 - 9W & 6 & 0 & -12 & -\frac{5}{2} & 0 \\ 0 & -15 + 9W & -6 & 0 & 12 & \frac{5}{2} & 0 \\ 0 & -14 + 9W & 96 - 54W & 0 & 12 & 4 & 0 \\ 0 & \frac{44}{3} - 9W & -28 + 18W & 0 & -12 & -3 & 0 \\ 0 & \frac{1}{3} & 34 - 18W & 0 & 0 & \frac{1}{2} & 0 \\ 0 & -1 & -102 + 54W & 0 & 0 & -\frac{3}{2} & 0 \end{pmatrix} \tag{3.82}$$

A similar matrix is obtained from $[\mathcal{C}, \tilde{21}']$:

$$\mathbf{A}_{\tilde{21},g} = \begin{pmatrix} & & & \vdots & & & \\ 0 & \frac{827}{135} & \frac{904}{45} & \frac{112}{45} & 0 & \frac{38}{9} - 3W & 2 \\ 0 & 0 & 0 & 2 - W & 0 & 1 & 0 \\ 0 & -5 + 3W & -2 & 0 & 4 & \frac{5}{6} & 0 \\ 0 & \frac{29}{3} - 6W & -30 + 18W & 0 & -8 & -\frac{13}{6} & 0 \\ 0 & \frac{14}{3} - 3W & -32 + 18W & 0 & -4 & -\frac{4}{3} & 0 \\ 0 & -\frac{13}{3} + 3W & 66 - 36W & 0 & 4 & \frac{11}{6} & 0 \\ 0 & -\frac{32}{3} + 6W & -72 + 36W & 0 & 8 & \frac{2}{3} & 0 \\ 0 & \frac{16}{3} - 3W & 36 - 18W & 0 & -4 & -\frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 34 - 18W & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix} \tag{3.83}$$

When solving the overdetermined system of equations, in both cases, there is a solution for the new coefficients depending again on one real parameter α; their values normalized to $\alpha = 6$ are

$$\alpha = 6 \,, \tag{3.84}$$

$$\beta_1 = -9 \,, \tag{3.85}$$

$$\beta_2 = -54 \,, \tag{3.86}$$

$$\gamma = -324 \,, \tag{3.87}$$

$$\delta = -207 + 108g(g-1) \,, \tag{3.88}$$

$$\epsilon = 648 - 324g(g-1) \,, \tag{3.89}$$

$$\zeta = -709 + 1656g(g-1) - 486g^2(g-1)^2 \,. \tag{3.90}$$

Since the coefficients α, $\beta := \beta_1$ and γ contain the terms which survive in the classical limit, they are not affected by the presence of the interaction, the values being identical with those found in the free case. The quantum effects are present in the sectors δ, ϵ, ζ, and by this reason their modification is a natural consequence of the interaction g; in comparison to the free case, their old values are shifted by a multiple of the factor $g(g-1)$, and clearly one recovers the free case not only when $g = 0$, but also with $g = 1$.

With this last result our original task is finally complete: A Casimir operator for the Calogero model with $N = 3$ particles for both quantum and classical cases has been found and expressed in terms of a basis of operators satisfying a W_3-algebra and applying Weyl ordering between those operators.

4 Conclusions

In the present work a Casimir operator for the Calogero model with three particles, and interaction potential inversely proportional to the square of the distance between two particles, has been found for the free and interaction cases (both quantum and classical). By construction, this Casimir operator is symmetric under particle permutations, and it is expressed in terms of operators satisfying a W_3-algebra; additionally, for the given set of basis operators a transformation was determined, so that the center-of-mass and total momentum of the system are decoupled, i.e. they do not appear explicitly in the expression for the Casimir operator.

After decoupling COM and total momentum, the case $N = 2$ particles is fully described by the usual $\mathfrak{sl}(2)$-algebra, and there is only one Casimir operator . However, with $N = 3$ particles the commutation relations experience important modifications, namely, there is a subalgebra $\mathfrak{sl}(2)$ determined by the level-2 operators B_{20}, B_{02} and B_{11}, represented in the Casimir operator by the sectors $\mathcal{C}_\alpha$, $\mathcal{C}_\delta$ and $\mathcal{C}_\zeta$ (this last one being the proper Casimir for $\mathfrak{sl}(2)$), which does not depend explicitly on the interaction g; and the level-3 operators B_{30}, B_{21}, B_{12}, B_{03} present in the sectors $\mathcal{C}_\beta$, $\mathcal{C}_\gamma$ and $\mathcal{C}_\epsilon$ generate non-linear products of level-2 operators (order $\hbar$) plus a constant term of order $\hbar^2$ proportional to the identity (*central extension*), being explicitly included in this term the interaction as an additive constant of the form $g(g-1)$.

A key role played the introduction of the so-called Weyl ordering of operators, with its help a simple and compact form for the Casimir operator as for the COM-decoupling transformations could be written. From the technical point of view, those Weyl-ordered products, although manifestly symmetric, can be very complex, especially when calculating with composed or non-pure products, and special formulae were derived for dealing with them.

Finally, there are important topics which must be considered in a future work:

* To get a better understanding of the role that group representation theory plays in defining the general structure of the algebra satisfied by the operators B_{kl}. Here we only focused on the commutators and their behavior in presence of the interaction coupling, but it is possible to apply a line of argumentation similar to [14], for example, by using the adjoint representation for the operators B_{kl}. This is also related to one more fundamental question: Is there any other Casimir operator for the case studied here? If yes, how many?

* Searching for general algorithms or combinatorial techniques for calculating composed Weyl-products in terms of pure ones; this will be extremely beneficial when applying the method here developed to problems with more than three bodies.

* Extension of the procedure to an arbitrary number N of particles. This can be optimized, as mentioned above, with the help of representation theory and proved techniques for Weyl ordering.

* Study of one simple three-body-problem in a scattering case and apply the formalism to perturbation theory.

The most important technical lesson, after all the algebraic work, relates to the necessity of developing standard programs or routines with some CAS (e.g. Mathematica™or Maple™); this will be extremely helpful in the study of problems with a large number of particles, especially if the operators satisfy a W_N-algebra and we work with Weyl ordering, like it was in the present case.

A Appendix

A.1 How to get a formula for $[B_{kl}, B_{mn}]$?

In the Section 2.2 it was mentioned the convenience of a general formula for evaluate $[B_{kl}, B_{mn}]$. Here we show the necessary procedure for getting a useful expression up to order $k+m-3; l+n-3$, good enough when considering the basis operators $B_{10}, \ldots, B_{03}$. Extensions to higher order are required only from $N = 5$ onwards, but from the operational point of view it is the same.

Let us begin remembering the typical commutator formulae from Quantum Mechanics [11]:

$$[A, BC] = [A, B]C + B[A, C] \ , \tag{A.1}$$

$$[AB, C] = A[B, C] + [A, C]B \ . \tag{A.2}$$

Knowing that ($\hbar = 1$)

$$[x, p] = i \ , \tag{A.3}$$

it is easy to find

$$[x^j, p] = ijx^{j-1} \tag{A.4}$$

and

$$[x, p^k] = ikp^{k-1} \ . \tag{A.5}$$

To simplify the algebraic calculations, we use formally

$$\frac{\partial}{\partial p}\left[x^j, p^k\right] = k\left[x^j p^{k-1} - p^{k-1}x^j\right] = k[x^j, p^{k-1}] \ . \tag{A.6}$$

Having determined the value of $[x^j, p^m]$ we can apply the last equation (A.6), and by integration (observing the given order in p and x!!) one finds the formulae (easy to prove by means of induction)

$$[x^j, p^k] = \sum_{\mu=1}^{k} \binom{k}{\mu} \frac{i^\mu j!}{(j-\mu)!} p^{k-\mu} x^{j-\mu} \ , \tag{A.7}$$

$$[x^j, p^k] = \sum_{\mu=1}^{j} \binom{j}{\mu} \frac{(-1)^{\mu+1} i^\mu k!}{(k-\mu)!} x^{j-\mu} p^{k-\mu} \ . \tag{A.8}$$

The full expression to be calculated (see Section 2.2) looks like this:

$$[B_{kl}, B_{mn}] = \frac{1}{4} \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} \left[x_\alpha^k p_\alpha^l + p_\alpha^l x_\alpha^k, x_\beta^m p_\beta^n + p_\beta^n x_\beta^m\right] \ . \tag{A.9}$$

Applying the formulae (A.1) and (A.2) for $[A, BC]$ and $[AB, C]$ we get:

$$[B_{kl}, B_{mn}] = \frac{1}{4}\sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}\left(-x_\alpha^k\left[x_\beta^m, p_\alpha^l\right]p_\beta^n + x_\beta^m\left[x_\alpha^k, p_\beta^n\right]p_\alpha^l + \right.$$
$$- x_\alpha^k p_\beta^n\left[x_\beta^m, p_\alpha^l\right] + \left[x_\alpha^k, p_\beta^n\right]x_\beta^m p_\alpha^l + p_\alpha^l x_\beta^m\left[x_\alpha^k, p_\beta^n\right] - \left[x_\beta^m, p_\alpha^l\right]p_\beta^n x_\alpha^k +$$
$$\left. + p_\alpha^l\left[x_\alpha^k, p_\beta^n\right]x_\beta^m - p_\beta^n\left[x_\beta^m, p_\alpha^l\right]x_\alpha^k\right) \ . \quad \text{(A.10)}$$

Since

$$\left[x_\alpha^k, p_\beta^n\right] = \delta_{\alpha\beta}\left[x_\beta^k, p_\beta^n\right] \ , \quad \text{(A.11)}$$

we finally obtain

$$[B_{kl}, B_{mn}] = \frac{1}{4}\sum_{\beta=1}^{N}\left\langle -x_\beta^k\left[x_\beta^m, p_\beta^l\right]p_\beta^n + x_\beta^m\left[x_\beta^k, p_\beta^n\right]p_\beta^l + \right.$$
$$- x_\beta^k p_\beta^n\left[x_\beta^m, p_\beta^l\right] + \left[x_\beta^k, p_\beta^n\right]x_\beta^m p_\beta^l + p_\beta^l x_\beta^m\left[x_\beta^k, p_\beta^n\right] - \left[x_\beta^m, p_\beta^l\right]p_\beta^n x_\beta^k +$$
$$\left. + p_\beta^l\left[x_\beta^k, p_\beta^n\right]x_\beta^m - p_\beta^n\left[x_\beta^m, p_\beta^l\right]x_\beta^k\right\rangle \ . \quad \text{(A.12)}$$

Up to now we have applied some formal identities and the algebra is still reasonable. In the next steps one must expand the commutators, but up to which order? The answer will depend on the number of particles in the system. For example, with $N = 3$ we should approximate

$$[x^j, p^k] \approx ijkx^{j-1}p^{k-1} - \frac{i^2}{2!}j(j-1)k(k-1)x^{j-2}p^{k-2} +$$
$$+ \frac{i^3}{3!}j(j-1)(j-2)k(k-1)(k-2)x^{j-3}p^{k-3} + \dots \ , \quad \text{(A.13)}$$

and, very important, the chosen order must be strictly observed and equally applied in all the commutators. After some lengthy and cumbersome calculations, the expression found is:

$$\Big[B_{kl}, B_{mn}\Big] \ = \ iC^1_{klmn}B_{k+m-1;l+n-1} \ + \ iC^3_{klmn}B_{k+m-3;l+n-3} \ + \ \mathcal{O}\left(B_{k+m-5;l+n-5}\right) \ . \quad \text{(A.14)}$$

The first factor C^1_{klmn} is given by

$$C^1_{klmn} = kn - lm \ , \quad \text{(A.15)}$$

while the second coefficient C^3_{klmn} has the form

$$C^3_{klmn} = \frac{1}{12}\Big\{ k(k-1)n(n-1)\left[(k-2+3m)(n-2+3l) - 3lm\right] +$$
$$\text{-}l(l-1)m(m-1)\left[(l-2+3n)(m-2+3k) - 3kn\right]\Big\} \ . \quad \text{(A.16)}$$

How can be verified the result (A.14)? Let us begin with a direct calculation of $[B_{12}, B_{34}]$:

$$[B_{12}, B_{34}] = \frac{1}{4}\sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}\left[x_\alpha^1 p_\alpha^2 + p_\alpha^2 x_\alpha^1, x_\beta^3 p_\beta^4 + p_\beta^4 x_\beta^3\right] =$$

$$= \frac{1}{4}\sum_{\beta=1}^{N}\Big\langle -x_\beta^1\left[x_\beta^3, p_\beta^2\right]p_\beta^4 + x_\beta^3\left[x_\beta^1, p_\beta^4\right]p_\beta^2 +$$

$$- x_\beta^1 p_\beta^4\left[x_\beta^3, p_\beta^2\right] + \left[x_\beta^1, p_\beta^4\right]x_\beta^3 p_\beta^2 + p_\beta^2 x_\beta^3\left[x_\beta^1, p_\beta^4\right] - \left[x_\beta^3, p_\beta^2\right]p_\beta^4 x_\beta^1 +$$

$$+ p_\beta^2\left[x_\beta^1, p_\beta^4\right]x_\beta^3 - p_\beta^4\left[x_\beta^3, p_\beta^2\right]x_\beta^1\Big\rangle \ . \quad \text{(A.17)}$$

In the following we can neglect the subindex β. Let us define

$$-A := x\left[x^3, p^2\right]p^4 + xp^4\left[x^3, p^2\right] + \left[x^3, p^2\right]p^4 x + p^4\left[x^3, p^2\right]x \ , \quad \text{(A.18)}$$

$$B := x^3\left[x, p^4\right]p^2 + \left[x, p^4\right]x^3 p^2 + p^2 x^3\left[x, p^4\right] + p^2\left[x, p^4\right]x^3 \ , \quad \text{(A.19)}$$

and with the help of the results

$$\left[x^j, p^2\right] = ij\left\langle x^{j-1}p + px^{j-1}\right\rangle =: ij\left\{x^{j-1}, p\right\} \ , \quad \text{(A.20)}$$

$$\left[x^2, p^k\right] = kj\left\langle xp^{k-1} + p^{k-1}x\right\rangle =: ik\left\{x, p^{k-1}\right\} \ . \quad \text{(A.21)}$$

a) **Calculations for** A. In this case, from the formula (A.21)

$$\left[x^3, p^2\right] = 3i\left(x^2 p + px^2\right) \ , \quad \text{(A.22)}$$

the expression to be simplified reads

$$- A = 3i\left\langle x^3 p^5 + xpx^2p^4 + xp^4x^2p + xp^5x^2 + x^2p^5x + px^2p^4x + p^4x^2px + p^5x^3\right\rangle$$

$$\vdots$$

$$= 3i\left\langle 4\left\{x^3, p^5\right\} - 2ix^2p^4 - x\left[x^2, p^4\right]p + 5ip^4x^2 - 5ix^2p^4 + p\left[x^2, p^4\right]x + 2ip^4x^2\right\rangle$$

$$\vdots$$

$$= 3i\left\langle 4\left\{x^3, p^5\right\} - 28i^2\left\{x, p^3\right\} - x\left[x^2, p^4\right]p + p\left[x^2, p^4\right]x\right\rangle$$

$$\vdots$$

$$= 3i\left\langle 4\left\{x^3, p^5\right\} - 28i^2\left\{x, p^3\right\} - 8i\left[x^2, p^4\right] + 12i^2\left\{x, p^3\right\}\right\rangle$$

$$\vdots$$

$$\Longrightarrow A = -12i\left\{x^3, p^5\right\} + 144i^3\left\{x, p^3\right\} \ . \quad \text{(A.23)}$$

b) **Calculations for** B. Remembering that

$$\left[x, p^k\right] = ikp^{k-1} \quad \text{(A.24)}$$

we get

$$\left[x, p^4\right] = 4ip^3 \ , \tag{A.25}$$

$$\begin{aligned}
\Longrightarrow -B &= 4i\left\langle x^3p^5 + p^3x^3p^2 + p^2x^3p^3 + p^5x^3\right\rangle \\
&= 4i\left\langle 2\left\{x^3, p^5\right\} + p^3\left[x^3, p^2\right] - \left[x^3, p^2\right]p^3\right\rangle \\
(\text{A.20}) \rightarrow \quad &= 4i\left\langle 2\left\{x^3, p^5\right\} + 3ip^3x^2p + 3ip^4x^2 - 3ix^2p^4 - 3ipx^2p^3\right\rangle \\
&\vdots \\
&= 4i\left\langle 2\left\{x^3, p^5\right\} + 6i^2p^3xp + 6ip^4x^2 - 6ix^2p^4 + 6i^2xp^3\right\rangle \\
&= 4i\left\langle 2\left\{x^3, p^5\right\} + 6i^2\left\{x, p^3\right\} - 6i\left[x^2, p^4\right]\right\rangle \\
&= 4i\left\langle 2\left\{x^3, p^5\right\} + 6i^2\left\{x, p^3\right\} - 24i^2\left\{x, p^3\right\}\right\rangle \\
&= 8i\left\{x^3, p^5\right\} - 72i^3\left\{x, p^3\right\} \ .
\end{aligned} \tag{A.26}$$

Then, adding the expressions for A and B:

$$A + B = -4i\left\{x^3, p^5\right\} + 144i^3\left\{x, p^3\right\} \ , \tag{A.27}$$

$$\begin{aligned}
\therefore [B_{12}, B_{34}] &= \frac{1}{4}\sum_{\beta=1}^{N}\left[-4i\left\{x_\beta^3, p_\beta^5\right\} + 72i^3\left\{x_\beta, p_\beta^3\right\}\right] \\
&= -2iB_{35} - 36iB_{13} \ .
\end{aligned} \tag{A.28}$$

Applying formulae:

$$C_{1234}^1 = 1 \cdot 4 - 2 \cdot 3 = -2 \ , \tag{A.29}$$

$$\begin{aligned}
C_{1234}^3 = \frac{1}{12}[-2 \cdot (2-1) \cdot 3 \cdot (3-1) \cdot [(2-2+3\cdot 4) \cdot (3-2+3\cdot 1) - 3 \cdot 1 \cdot 4] \\
= -\frac{432}{12} = -36 \ ,
\end{aligned} \tag{A.30}$$

$$(\text{A.14}) \Longrightarrow [B_{12}, B_{34}] = -2iB_{35} - 36iB_{13} \ . \tag{A.31}$$

Repeating the direct procedure for $[B_{14}, B_{23}]$:

$$\begin{aligned}
[B_{14}, B_{23}] = \frac{1}{4}\sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}\left[x_\alpha^1p_\alpha^4 + p_\alpha^4x_\alpha^1, x_\beta^2p_\beta^3 + p_\beta^3x_\beta^2\right] = \\
= \frac{1}{4}\sum_{\beta=1}^{N}\left\langle -x_\beta\left[x_\beta^2, p_\beta^4\right]p_\beta^3 + x_\beta^2\left[x_\beta, p_\beta^3\right]p_\beta^4 + \right. \\
- x_\beta p_\beta^3\left[x_\beta^2, p_\beta^4\right] + \left[x_\beta, p_\beta^3\right]x_\beta^2p_\beta^4 + p_\beta^4x_\beta^2\left[x_\beta, p_\beta^3\right] - \left[x_\beta^2, p_\beta^4\right]p_\beta^3x_\beta + \\
\left. + p_\beta^4\left[x_\beta, p_\beta^3\right]x_\beta^2 - p_\beta^3\left[x_\beta^2, p_\beta^4\right]x_\beta\right\rangle \ ,
\end{aligned} \tag{A.32}$$

$$-A := x\left[x^2,p^4\right]p^3 + xp^3\left[x^2,p^4\right] + \left[x^2,p^4\right]p^3x + p^3\left[x^2,p^4\right]x\ , \tag{A.33}$$

$$B := x^2\left[x,p^3\right]p^4 + \left[x,p^3\right]x^2p^4 + p^4x^2\left[x,p^3\right] + p^4\left[x,p^3\right]x^2\ . \tag{A.34}$$

After the algebra one gets:

$$A = -16i\left\{x^2,p^6\right\} + 240i^3p^4\ , \tag{A.35}$$

$$B = 6i\left\{x^2,p^6\right\} - 48i^3p^4\ , \tag{A.36}$$

$$\begin{aligned} A+B \Longrightarrow [B_{14},B_{23}] &= \frac{1}{4}\sum_{\beta=1}^{N}\left\langle -10i\left\{x_\beta^2,p_\beta^6\right\} - 192ip_\beta^4\right\rangle \\ &= -5iB_{26} - 48iB_{04}\ . \end{aligned} \tag{A.37}$$

Now using the formulae

$$C^1_{1423} = 1\cdot 3 - 4\cdot 2 = -5\ , \tag{A.38}$$

$$\begin{aligned} C^3_{1423} = \frac{1}{12}&[1\cdot(1-1)\cdot 3\cdot(3-1)\cdot[(1-2+3\cdot 2)\cdot(3-2+3\cdot 4) - 3\cdot 4\cdot 2] \\ &- 4\cdot(4-1)\cdot 2\cdot(2-1)\cdot[(4-2+3\cdot 3)\cdot(2-2+3\cdot 1) - 3\cdot 1\cdot 3] = -\frac{576}{12} = -48\ . \end{aligned} \tag{A.39}$$

$$\Longrightarrow [B_{14},B_{23}] = -5iB_{26} - 48iB_{04}\ . \tag{A.40}$$

One last example for verifying the formulae (A.14), (A.15) and (A.16):

$$\begin{aligned} [B_{15},B_{24}] = \frac{1}{4}\sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}&\left[x_\alpha^1p_\alpha^5 + p_\alpha^5x_\alpha^1, x_\beta^2p_\beta^4 + p_\beta^4x_\beta^2\right] = \\ &= \frac{1}{4}\sum_{\beta=1}^{N}\left\langle -x_\beta\left[x_\beta^2,p_\beta^5\right]p_\beta^4 + x_\beta^2\left[x_\beta,p_\beta^4\right]p_\beta^5 + \right. \\ &- x_\beta p_\beta^4\left[x_\beta^2,p_\beta^5\right] + \left[x_\beta,p_\beta^4\right]x_\beta^2p_\beta^5 + p_\beta^5x_\beta^2\left[x_\beta,p_\beta^4\right] - \left[x_\beta^2,p_\beta^5\right]p_\beta^4x_\beta + \\ &\left. + p_\beta^5\left[x_\beta,p_\beta^4\right]x_\beta^2 - p_\beta^4\left[x_\beta^2,p_\beta^5\right]x_\beta\right\rangle\ , \end{aligned} \tag{A.41}$$

$$-A := x\left[x^2,p^5\right]p^4 + xp^4\left[x^2,p^5\right] + \left[x^2,p^5\right]p^4x + p^4\left[x^2,p^5\right]x\ , \tag{A.42}$$

$$B := x^2\left[x,p^4\right]p^5 + \left[x,p^4\right]x^2p^5 + p^5x^2\left[x,p^4\right] + p^5\left[x,p^4\right]x^2\ . \tag{A.43}$$

As a result of the algebra we get:

$$A = -20i\left\{x^2,p^8\right\} + 560i^3p^6\ , \tag{A.44}$$

$$B = 8i\left\{x^2,p^8\right\} - 120i^3p^6\ , \tag{A.45}$$

$$\begin{aligned} A+B \Longrightarrow [B_{15},B_{24}] &= \frac{1}{4}\sum_{\beta=1}^{N}\left\langle -12i\left\{x_\beta^2,p_\beta^8\right\} - 440ip_\beta^6\right\rangle \\ &= -6iB_{28} - 110iB_{06} \Longrightarrow C^3_{1524} = -110\ . \end{aligned} \tag{A.46}$$

Now using the formula (A.14), (A.15) and (A.16):

$$C^1_{1524} = 1 \cdot 4 - 5 \cdot 2 = -6 \ , \tag{A.47}$$

$$\begin{aligned} C^3_{1524} &= \frac{1}{12}[1 \cdot (1-1) \cdot 4 \cdot (4-1)[(1-2+3 \cdot 2) \cdot (4-2+3 \cdot 5) - 3 \cdot 5 \cdot 2] \\ &- 5 \cdot (5-1) \cdot 2 \cdot (2-1) \cdot [(5-2+3 \cdot 4) \cdot (2-2+3 \cdot 1) - 3 \cdot 1 \cdot 4]] = -\frac{1320}{12} = -110 \end{aligned} \tag{A.48}$$

$$\text{(A.14)} \Longrightarrow [B_{15}, B_{24}] = -6iB_{28} - 110iB_{06} \ . \tag{A.49}$$

We repeat one more time: the present work aims to find a Casimir operator in the context of Calogero models, with the help of the B_{kl} operators. Since the existence of a Casimir requires the vanishing of its commutator with the rest of operators, we have two possibilities when defining a basis of operators for the case $N = 3$:

* Either we calculate directly each required commutator, by expanding, collecting terms, etc;

* or we determine a general expression for the commutator $[B_{kl}, B_{mn}]$, good enough when dealing with operators B_{kl}, $k + l \leq N = 3$.

Clearly, the second option suits better to our present needs, especially when it minimizes the rapidly increasing length of the algebraic calculations.

A.2 Dunkl Operators π_k

We prove an important property of the Dunkl operators associated to the Calogero Model with Hamiltonian given by

$$H = \frac{1}{2}\sum_{\alpha} p_\alpha^2 + \sum_{\alpha<\beta} \frac{g(g-1)}{(x_\alpha - x_\beta)^2} \ . \tag{A.50}$$

Their definition is

$$\pi_\alpha = p_\alpha + i \sum_{\mu(\neq\alpha)} \frac{g}{x^\alpha - x^\mu} s_{\alpha\mu}, \text{ for } \alpha, \mu = 1 \dots N \ , \tag{A.51}$$

or in terms of derivatives:

$$\mathcal{D}_\alpha = \partial_\alpha - \sum_{\mu(\neq\alpha)} \frac{g}{x^\alpha - x^\mu} s_{\alpha\mu} \ . \tag{A.52}$$

"$s_{\alpha\nu}$" is the 2-particle exchange operator with the following properties ([1, 2]):

$$s_{ab} = (s_{ab})^{-1} = s_{ab}^\dagger = s_{ba} \ , \tag{A.53}$$

$$[s_{ab}, s_{cd}] = 0, \ a \neq b \neq c \neq d \ , \tag{A.54}$$

$$s_{ab} s_{bc} = s_{ac} s_{ab}, \ a \neq b \neq c \ . \tag{A.55}$$

Those operators act on the space of all symmetric functions under exchange of two particles [1]. In that case, they satisfy the important result

$$[\pi_a, \pi_b] = 0, \ \forall a, b = 1 \dots N \ , \tag{A.56}$$

which will be proved by direct calculation. Let us start by defining

$$x_{ab} := x_a - x_b \ . \tag{A.57}$$

Clearly

$$p_\mu \left(x_{ab}^{-n}\right) = i \cdot n \left(\delta_{a\mu} - \delta_{b\mu}\right) x_{ab}^{-n-1} \ , \tag{A.58}$$

$$\left[p_\mu, x_{ab}^{-1}\right] = i \left(\delta_{a\mu} - \delta_{b\mu}\right) x_{ab}^{-2} \ . \tag{A.59}$$

Expanding the commutator $[\pi_i, \pi_j]$:

$$\begin{aligned}
[\pi_a, \pi_b] = & \left[p_a + i \sum_{\mu(\neq a)} \frac{g}{x_{a\mu}} s_{a\mu}, p_b + i \sum_{\nu(\neq b)} \frac{g}{x_{b\nu}} s_{b\nu} \right] \\
& = [p_a, p_b] + i \sum_{\nu(\neq b)} \left[p_a, \frac{g}{x_{b\nu}} s_{b\nu} \right] + i \sum_{\mu(\neq a)} \left[\frac{g}{x_{a\mu}} s_{a\mu}, p_b \right] + \\
& \qquad + i^2 g^2 \left[\frac{g}{x_{a\mu}} s_{a\mu}, , \frac{g}{x_{b\nu}} s_{b\nu} \right] \ .
\end{aligned} \tag{A.60}$$

Now we separately analyze each commutator.

a) It is evident the vanishing of the first term in (A.58)

$$[p_a, p_b] = 0 \ . \tag{A.61}$$

b) The second member of (A.58) is

$$ig \sum_{\nu(\neq b)} \left[p_a, \frac{1}{x_{b\nu}} s_{b\nu}\right] \psi = ig \left[p_a, \frac{1}{x_{ba}} s_{ba}\right] \psi + ig \sum_{\nu(\neq a,b)} \left[p_a, \frac{1}{x_{b\nu}} s_{b\nu}\right] \psi \ . \tag{A.62}$$

The term without sum over ν:

$$\left[p_a, \frac{s_{ba}}{x_{ba}}\right] \psi = \left[p_a, \frac{1}{x_{ba}}\right] s_{ba} \psi + \frac{1}{x_{ba}} [p_a, s_{ba}] \psi = -\frac{i}{x_{ba}^2} \psi - \frac{(p_b - p_a)}{x_{ba}} \psi \ . \tag{A.63}$$

The second term vanishes, because $a \neq b$ and $\nu \neq a$:

$$ig \sum_{\nu(\neq a,b)} \left[p_a, \frac{1}{x_{b\nu}} s_{b\nu}\right] \psi = 0 \ . \tag{A.64}$$

Then:

$$ig \sum_{\nu(\neq b)} \left[p_a, \frac{1}{x_{b\nu}} s_{b\nu}\right] = -\frac{i^2 g}{x_{ba}^2} + \frac{ig\,(p_a - p_b)}{x_{ba}} \ . \tag{A.65}$$

c) Proceeding with the third term in the same way:

$$ig \sum_{\mu(\neq a)} \left[\frac{1}{x_{a\mu}} s_{a\mu}, p_b\right] \psi = ig \left[\frac{1}{x_{ab}} s_{ab}, p_b\right] \psi + ig \sum_{\mu \neq a \neq b} \left[\frac{1}{x_{a\mu}} s_{a\mu}, p_b\right] \psi \ . \tag{A.66}$$

The last commutator in (A.66) vanishes due to the fact $\mu \neq a \neq b$. For the first commutator:

$$\left[\frac{1}{x_{ab}} s_{ab}, p_b\right] \psi = \frac{1}{x_{ab}} [s_{ab}, p_b] \psi + \left[\frac{1}{x_{ab}}, p_b\right] s_{ab} \psi = \frac{(p_a - p_b)}{x_{ab}} \psi + \frac{i^2}{x_{ab}^2} \psi \ , \tag{A.67}$$

$$\Longrightarrow ig \sum_{\mu(\neq a)} \left[\frac{1}{x_{a\mu}} s_{a\mu}, p_b\right] = \frac{ig\,(p_a - p_b)}{x_{ab}} + \frac{i^2 g}{x_{ab}^2} \ . \tag{A.68}$$

d) For the last commutator, the trick consists in separating the sums according to the different possibilities:

$$\begin{aligned} \sum_{\mu \neq a} \sum_{\nu \neq b} \left[\frac{1}{x_{a\mu}} s_{a\mu}, \frac{1}{x_{b\nu}} s_{b\nu}\right] &= \sum_{\mu \neq a} \left[\frac{1}{x_{a\mu}} s_{a\mu}, \frac{1}{x_{ba}} s_{ba}\right] + \sum_{\mu \neq a} \sum_{\nu \neq a \neq b} \left[\frac{1}{x_{a\mu}} s_{a\mu}, \frac{1}{x_{b\nu}} s_{b\nu}\right] \\ &= \left[\frac{1}{x_{ab}} s_{ab}, \frac{1}{x_{ba}} s_{ba}\right] + \sum_{\mu \neq a \neq b} \left[\frac{1}{x_{a\mu}} s_{a\mu}, \frac{1}{x_{ba}} s_{ba}\right] + \sum_{\nu \neq a \neq b} \left[\frac{1}{x_{ab}} s_{ab}, \frac{1}{x_{b\nu}} s_{b\nu}\right] + \\ &\quad + \sum_{\mu = \nu \neq a \neq b} \left[\frac{1}{x_{a\mu}} s_{a\mu}, \frac{1}{x_{b\mu}} s_{b\mu}\right] + \sum_{\mu \neq \nu \neq a \neq b} \left[\frac{1}{x_{a\mu}} s_{a\mu}, \frac{1}{x_{b\nu}} s_{b\nu}\right] \ . \end{aligned} \tag{A.69}$$

Emphasizing the fact we apply those π_k operators to functions belonging to $res(\pi_k)$, the last expressions become successively

$$\sum_{\mu\neq a\neq b}\left[\frac{1}{x_{a\mu}}s_{a\mu},\frac{1}{x_{ba}}s_{ba}\right]=\sum_{\mu\neq a\neq b}\left(\frac{1}{x_{a\mu}x_{b\mu}}-\frac{1}{x_{ba}x_{b\mu}}\right)\ , \tag{A.70}$$

$$\sum_{\nu\neq a\neq b}\left[\frac{1}{x_{ab}}s_{ab},\frac{1}{x_{b\nu}}s_{b\nu}\right]=\sum_{\nu\neq a\neq b}\left(\frac{1}{x_{ab}x_{a\nu}}-\frac{1}{x_{b\nu}x_{a\nu}}\right)\ , \tag{A.71}$$

$$\sum_{\mu=\nu\neq a\neq b}\left[\frac{1}{x_{a\mu}}s_{a\mu},\frac{1}{x_{b\mu}}s_{b\mu}\right]=\sum_{\mu=\nu\neq a\neq b}\left(\frac{1}{x_{a\mu}x_{ba}}-\frac{1}{x_{b\mu}x_{ab}}\right)\ , \tag{A.72}$$

$$\sum_{\mu\neq\nu\neq a\neq b}\left[\frac{1}{x_{a\mu}}s_{a\mu},\frac{1}{x_{b\nu}}s_{b\nu}\right]=0\ . \tag{A.73}$$

Adding the equations (A.69), (A.70), (A.71) and (A.72), we obtain:

$$\sum_{\mu\neq a}\sum_{\nu\neq b}\left[\frac{1}{x_{a\mu}}s_{a\mu},\frac{1}{x_{b\nu}}s_{b\nu}\right]=0\ . \tag{A.74}$$

Inserting the results (A.61), (A.65), (A.68), (A.74) in (A.58), we arrive to the vanishing of the commutator $[\pi_a,\pi_b]$:

$$[\pi_a,\pi_b]=0,\ \forall a,b=1\ldots N\ . \tag{A.75}$$

A.3 COM-Decoupling and Commutators

Let us recall the transformation found in Section 2.4 for COM-decoupling:

* Level 1 operators:

$$\left.\begin{array}{rcl} (10') &=& (10) \\ (01') &=& (01) \\ (00') &=& (00) = N = 3 \end{array}\right\} . \tag{A.76}$$

* Level 2 operators:

$$\left.\begin{array}{rcl} (20') &=& (20) - \frac{1}{3}(10|10) \\ (11') &=& (11) - \frac{1}{3}(10|01) \\ (02') &=& (02) - \frac{1}{3}(01|01) \end{array}\right\} . \tag{A.77}$$

* Level 3 operators:

$$\left.\begin{array}{rcl} (30') &=& (30) - (20|10) + \frac{2}{9}(10|10|10) \\ (21') &=& (21) - \frac{1}{3}(20|01) - \frac{2}{3}(11|10) + \frac{2}{9}(10|10|01) \\ (12') &=& (12) - \frac{2}{3}(11|01) - \frac{1}{3}(10|02) + \frac{2}{9}(10|01|01) \\ (03') &=& (03) - (02|01) + \frac{2}{9}(01|01|01) \end{array}\right\} . \tag{A.78}$$

Because of their importance when dealing with the interaction Casimir operator, we show in a detailed way the calculation for $[30', 03']$:

$$\begin{aligned} [30', 03'] = [30, 03] - [30, (02|01)] + \frac{2}{9} \cdot [30, (01|01|01)] \\ - [(20|10), 03] + [(20|10), (02|01)] - \frac{2}{9} \cdot [(20|10), (01|01|01)] \\ + \frac{2}{9} \cdot [(10|10|10), 03] - \frac{2}{9} \cdot [(10|10|10), (02|01)] + \frac{4}{81} \cdot [(10|10|10), (01|01|01)] \ , \end{aligned} \tag{A.79}$$

Useful formulae for the expansions:

$$(a|b|(c|d)) = (a|b|c|d) + (a|R^b_{cd}) + (b|R^a_{cd}) + \ + \frac{1}{12}([a, c]|[b, d]) + \frac{1}{12}([a, d]|[b, c]) \ , \tag{A.80}$$

$$(a|(b|c|d)) = (a|b|c|d) + (b|R^a_{cd}) + (c|R^a_{bd}) + (d|R^a_{bc}) \ , \tag{A.81}$$

$$[30, 03] = 9i(22) + 3i(00) \ , \tag{A.82}$$

$$[30, (02|01)] = 6i(21|01) + 3i(20|02) \ , \tag{A.83}$$

$$[30, (01|01|01)] = 9i(20|01|01) \ , \tag{A.84}$$

$$[(20|10),03] = 6i(12|10) + 3i(20|02) \ , \tag{A.85}$$

$$[(20|10),(01|01|01)] = 2i(10100101) + Ni(200101) + \frac{N^2 i^3}{3} \ , \tag{A.86}$$

$$[(20|10),(02|01)] = 4i(11|10|01) + 2i(20|01|01) + 2i(10|10|02) + Ni(20|02) + \frac{6i^3}{N} \ , \tag{A.87}$$

$$[(10|10|10),03] = 3i(20|10|10) \ , \tag{A.88}$$

$$-\frac{6}{N^3}[(10|10|10),(02|01)] = -\frac{36i}{N^3}(10|10|01|01) - \frac{18i}{N^2}(10|10|02) - \frac{6i^3}{N} \ , \tag{A.89}$$

$$\frac{4}{N^4}[(10|10|10),(01|01|01)] = \frac{36i}{N^3}(10|10|01|01) + \frac{6i}{N} \ . \tag{A.90}$$

Adding the equation, and collecting similar terms, one gets

$$\begin{aligned}[30',03'] = 9i(22) + 3i(00) - \frac{18i}{N}(21|01) - \frac{9i}{N}(20|02) - \frac{18i}{N}(12|10) + \\ + \frac{18i}{N^2}(20|01|01) + \frac{36i}{N^2}(11|10|01) + \frac{18i}{N^2}(10|01|02) - \frac{36i}{N^3}(10|10|01|01) \ .\end{aligned} \tag{A.91}$$

The expression for B_{22} with COM must be inserted in the last equation:

$$\begin{aligned}B_{22} = \frac{2}{3}(21|01) + \frac{1}{6}(20|02) + \frac{2}{3}(12|10) + \frac{1}{3}(11|11) + \\ - \frac{1}{6}(20|01|01) - \frac{2}{3}(11|10|01) - \frac{1}{6}(10|10|02) + \frac{1}{6}(10|10|01|01) - \frac{3}{2} \ .\end{aligned} \tag{A.92}$$

Collecting terms, replacing $N = 3$, etc., the result is:

$$\begin{aligned}i^{-1}[30',03'] = -\frac{3}{2}(20|02) + 3(11|11) + \frac{1}{2}(20|01|01) - 2(11|10|01) + \frac{1}{2}(10|10|02) + \\ + \frac{1}{6}(10|10|01|01) - \frac{9}{2} \ .\end{aligned} \tag{A.93}$$

This result is given in the old COM-basis, but we need to express it in terms of the new primed basis. We proceed by direct calculation of

$$\begin{aligned}(20'|02') = \left(20 - \frac{1}{3}(10|10)\,\middle|\, 02 - \frac{1}{3}(01|01)\right) \\ = (20|02) - \frac{1}{3}(20|(01|01)) - \frac{1}{3}(02|(10|10)) + \frac{1}{9}((10|10)|(01|01)) \ .\end{aligned} \tag{A.94}$$

Applying the already known formulae for composed Weyl-ordered products, for example

$$(20|(01|01)) = (20|01|01) + R^{20}_{01,01} = (20|01|01) + i^2 \ , \tag{A.95}$$

we arrive to

$$(20'|02') = (20|02) - \frac{1}{3}(20|01|01) - \frac{1}{3}(10|10|02) + \frac{1}{9}(10|10|01|01) - \frac{i^2}{6} \ . \tag{A.96}$$

Similarly:

$$(11'|11') = \left(11 - \frac{1}{3}(10|01)\right| 11 - \frac{1}{3}(10|01)\Big)$$
$$= (11|11) - \frac{2}{3}(11|(10|01)) + \frac{1}{9}((10|01)|(10|01)) \ , \quad (A.97)$$

$$\Longrightarrow (11'|11') = (11|11) - \frac{2}{3}(11|10|01) + \frac{1}{9}(10|10|01|01) + \frac{i^2}{12} \ . \quad (A.98)$$

Combining the results:

$$-\frac{3}{2}(20'|02') + 3(11'|11') = -\frac{3}{2}(20|02) + 3(11|11) +$$
$$+ \frac{1}{2}(20|01|01) - 2(11|10|01) + \frac{1}{2}(10|10|02) + \frac{1}{6}(10|10|01|01) + \frac{i^2}{2} \ . \quad (A.99)$$

Replacing in (A.93), and collecting constant terms, we get the desired result:

$$[30', 03'] = -\frac{3}{2}(20'|02') + 3(11'|11') - 4 \ . \quad (A.100)$$

For the case of $[21', 12']$ the procedure is the same:

$$[21', 12'] = [21, 12] - \frac{2}{3}[21, (11|01)] - \frac{1}{3}[21, (10|02)] + \frac{2}{9}[21, (10|01|01)] +$$

$$- \frac{1}{3}[(20|01), 12] + \frac{2}{9}[(20|01), |(11|01)] + \frac{1}{9}[(20|01), (10|02)] - \frac{2}{27}[(20|01), (10|01|01)] +$$

$$- \frac{2}{3}[(11|10), 12] + \frac{4}{9}[(11|10), |(11|01)] + \frac{2}{9}[(11|10), (10|02)] - \frac{4}{27}[(11|10), (10|01|01)] +$$

$$- \frac{2}{9}[(10|10|01), 12] - \frac{4}{27}[(10|10|01), |(11|01)] - \frac{2}{27}[(10|10|01), (10|02)] +$$

$$+ \frac{4}{81}[(10|10|01), (10|01|01)] \ . \quad (A.101)$$

Expanding each product, collecting terms and paying attention to the constants:

$$i^{-1}[21', 12'] = \frac{5}{6}(20|02) - \frac{1}{3}(11|11) - \frac{5}{18}(20|01|01) + \frac{2}{9}(11|10|01) - \frac{5}{18}(10|10|02) +$$
$$+ \frac{1}{18}(10|10|01|01) + \frac{3}{2} \ . \quad (A.102)$$

Before we continue, we note the difference between this last result and (A.93), because the products $(20|02)$ and $(11|11)$ exhibit different prefactors:

$$
\begin{aligned}
i^{-1}[30', 03'] &= -\tfrac{1}{6}(20|02) + 3(11|11) + \ldots \\
i^{-1}[21', 12'] &= \tfrac{5}{6}(20|02) - \tfrac{1}{3}(11|11) + \ldots
\end{aligned} \quad . \tag{A.103}
$$

Again, the equation (A.102) must be expressed in terms of the primed basis, etc. The final expression then reads

$$
i^{-1}[21', 12'] = \frac{5}{6}(20'|02') - \frac{1}{3}(11'|11') + \frac{4}{3} \ . \tag{A.104}
$$

The rest of the commutators are found using the same method: firstly, a direct calculation by means of the definition, then a second calculation for expressing the result in terms of the primed basis. The only difficult point has to do with the composed Weyl-ordered products and the constants appearing when reducing to lower order terms.

A.4 Casimir Operator for $N = 2$ particles

We work in the following basis:

$$\{B_{10}; B_{01}; B_{20}; B_{02}; B_{11}\} \ . \tag{A.105}$$

The operator $B_{00} = N = 2$ is only a constant, which does not affect the Weyl-ordered products.

A.4.1 Commutators

Note: The factor $\pm i$ in the results was omitted to simplify the writing.

$[B_{kl}, B_{mn}]$	B_{10}	B_{01}	B_{20}	B_{02}	B_{11}
B_{10}	0	N	0	$2B_{01}$	B_{10}
B_{01}	N	0	$-2B_{10}$	0	$-B_{01}$
B_{20}	0	$2B_{10}$	0	$4B_{11}$	$2B_{20}$
B_{02}	$-2B_{01}$	0	$-4B_{11}$	0	$-2B_{02}$
B_{11}	$-B_{10}$	B_{01}	$-2B_{20}$	$2B_{02}$	0

. (A.106)

A.4.2 Ansatz

$$\mathcal{C}_{22} = A(20|02)+B(11|11)+C(20|01|01)+D(11|10|01)+E(10|10|02)+F(10|10|01|01) \ . \tag{A.107}$$

A.4.3 Calculations

Note: $\pm i$ factors are omitted!

$$\begin{array}{rcl} [(20|02)\,,10] &=& -2\,(20|01) \\ [(11|11)\,,10] &=& -2\,(11|10) \\ [(20|01|01)\,,10] &=& -2N\,(20|01) \\ [(11|10|01)\,,10] &=& -\,(10|10|01) - N\,(11|10) \\ [(10|10|02)\,,10] &=& -2\,(10|10|01) \\ [(10|10|01|01)\,,10] &=& -2N\,(10|10|01) \end{array} , \tag{A.108}$$

$$\begin{array}{rcl} [(20|02)\,,01] &=& 2\,(10|02) \\ [(11|11)\,,01] &=& 2\,(11|01) \\ [(20|01|01)\,,01] &=& 2\,(10|01|01) \\ [(11|10|01)\,,01] &=& (10|01|01) + N\,(11|01) \\ [(10|10|02)\,,01] &=& 2N\,(10|02) \\ [(10|10|01|01)\,,01] &=& 2N\,(10|01|01) \end{array} , \tag{A.109}$$

$$\begin{array}{rcl} [(20|02)\,,20] & = & -4\,(20|11) \\ [(11|11)\,,20] & = & -4\,(20|11) \\ [(20|01|01)\,,20] & = & -4\,(20|10|01) \\ [(11|10|01)\,,20] & = & -2\,(20|10|01) - 2\,(11|10|10) \\ [(10|10|02)\,,20] & = & -4\,(11|10|10) \\ [(10|10|01|01)\,,20] & = & -4\,(10|10|10|01) \end{array} , \tag{A.110}$$

$$\begin{array}{rcl} [(20|02)\,,02] & = & 4\,(11|02) \\ [(11|11)\,,02] & = & 4\,(11|02) \\ [(20|01|01)\,,02] & = & 4\,(11|01|01) \\ [(11|10|01)\,,02] & = & 2\,(10|02|01) + 2\,(11|01|01) \\ [(10|10|02)\,,02] & = & 4\,(10|02|01) \\ [(10|10|01|01)\,,02] & = & 4\,(10|01|01|01) \end{array} . \tag{A.111}$$

The equations obtained can be summarized in the next expression:

$$\begin{pmatrix} 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 1 & 0 & 4 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} . \tag{A.112}$$

Using Maple$^{\text{TM}}$, we have a general solution given by

$$\begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \end{pmatrix} = \alpha \begin{pmatrix} -2 \\ 2 \\ 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} . \tag{A.113}$$

Then, the Casimir operator for $N = 2$ and including center-of-mass takes this form:

$$\mathcal{C}_{22} = \alpha \Big[-2(20|02) + 2(11|11) + (20|01|01) - 2(11|10|01) + (10|10|02) \Big] . \tag{A.114}$$

Just for completeness, we calculate the commutator $[\mathcal{C}_{22}, 11]$:

$$\begin{array}{rcl} [(20|02)\,,11] & = & 2\,(20|02) - 2\,(20|02) = 0 \\ [(11|11)\,,11] & = & 0 \\ [(20|01|01)\,,11] & = & 2\,(20|01|01) - 2\,(20|01|01) = 0 \\ [(11|10|01)\,,11] & = & (11|10|01) - (11|10|01) = 0 \\ [(10|10|02)\,,11] & = & 2\,(10|10|02) - 2\,(10|10|02) = 0 \\ [(10|10|01|01)\,,11] & = & 2\,(10|10|01|01) - 2\,(10|10|01|01) = 0 \end{array} \quad . \tag{A.115}$$

So, the technique applied for $N = 2$ can be extended to $N = 3$ in a similar way, with two differences:

* The operator basis includes now B_{30}, B_{03}, B_{21} and B_{12}:

* The terms (21|01) and (12|10) will appear in the Casimir Operator, and their commutator with level-3 operators will generate composed Weyl-ordered products, increasing the order of the system of equations, but also producing terms that are not present in the original Ansatz.

A.5 Classical Poisson Casimir

When considering the limit $\hbar \to 0$ the contributions $\mathcal{C}_\delta$, $\mathcal{C}_\epsilon$ and $\mathcal{C}_\zeta$ will disappear, but the other terms will remain.

The main difference in the case of the Poisson bracket lies in the fact that all the variables commute between themselves. Using the definition

$$\{A, B\} = \sum_{\alpha=1}^{N} \left(\frac{\partial A}{\partial x_\alpha} \frac{\partial B}{\partial p_\alpha} - \frac{\partial A}{\partial p_\alpha} \frac{\partial B}{\partial x_\alpha} \right) , \tag{A.116}$$

it is immediately clear:

$$\{x_\mu, p_\nu\} = \delta_{\mu\nu} . \tag{A.117}$$

As a practical consequence of commutativity, all the constants disappear when reordering the Weyl-ordered products. But more important: there are no more composed Weyl-ordered products!!

We show one typical example, namely, we repeat the calculation for $[30, 03]$:

$$B_{30} := 30 = \sum_{\mu}^{3} x_\mu^3 , \tag{A.118}$$

$$B_{03} := 03 = \sum_{\nu}^{3} p_\nu^3 , \tag{A.119}$$

$$\{30, 03\} = \sum_{\mu,\nu}^{3} \{x_\mu^3, p_\nu^3\} = \sum_{\mu,\nu}^{3} \{x_\mu^3, p_\nu\} p_\nu^2 + \sum_{\mu,\nu}^{3} p_\nu \{x_\mu^3, p_\nu^2\} . \tag{A.120}$$

Applying bracket properties:

$$\{x_\mu^3, p_\nu\} p_\nu^2 = 3\delta_{\mu\nu} x_\mu^2 p_\nu^2 , \tag{A.121}$$

$$\begin{aligned} \{x_\mu^3, p_\nu^2\} &= \{x_\mu^3, p_\nu\} p_\nu + p_\nu \{x_\mu^3, p_\nu\} \\ &= 6\delta_{\mu\nu} x_\mu^2 p_\nu . \end{aligned} \tag{A.122}$$

Inserting those results in (A.120):

$$\{30, 03\} = \sum_{\mu,\nu}^{3} 9\delta_{\mu\nu} x_\mu^2 p_\nu^2 = 9 \left(\sum_{\mu}^{3} x_\mu^2 p_\mu^2 \right) . \tag{A.123}$$

By definition, the operator B_{22} in the classical case reads

$$B_{22} = \frac{1}{2} \sum_{\mu}^{3} \left(x_\mu^2 p_\mu^2 + p_\mu^2 x_\mu^2 \right) = \sum_{\mu}^{3} x_\mu^2 p_\mu^2 , \tag{A.124}$$

and it is evident

$$\{30, 03\} = 9(22) . \tag{A.125}$$

Similarly for the second most important commutator:

$$B_{21} := 21 = \frac{1}{2}\sum_{\mu=1}^{3}\left(x_\mu^2 p_\mu + p_\mu x_\mu^2\right) = \sum_{\mu=1}^{3} x_\mu^2 p_\mu \ , \tag{A.126}$$

$$B_{12} := 12 = \frac{1}{2}\sum_{\mu=1}^{3}\left(x_\mu p_\mu^2 + p_\mu^2 x_\mu\right) = \sum_{\mu=1}^{3} x_\mu p_\mu^2 \ , \tag{A.127}$$

$$\{21, 12\} = \sum_{\mu,\nu=1}^{3}\left\{x_\mu^2 p_\mu, x_\nu p_\nu^2\right\} = \sum_{\mu,\nu=1}^{3}\left\{x_\mu^2 p_\mu, x_\nu\right\} p_\nu^2 + \sum_{\mu,\nu=1}^{3} x_\nu \left\{x_\mu^2 p_\mu, p_\nu^2\right\} \ . \tag{A.128}$$

Bracket properties:

$$\left\{x_\mu^2 p_\mu, x_\nu\right\} p_\nu^2 = x_\mu^2 \left\{p_\mu, x_\nu\right\} p_\nu^2 = -\delta_{\mu\nu} x_\mu^2 p_\nu^2 \ , \tag{A.129}$$

$$x_\nu \left\{x_\mu^2 p_\mu, p_\nu^2\right\} = x_\nu \left\{x_\mu^2, p_\nu^2\right\} p_\mu \tag{A.130}$$

$$= 4\delta_{\mu\nu} x_\mu x_\nu p_\mu p_\nu \ . \tag{A.131}$$

(A.129) and (A.130) in (A.128):

$$\{21, 12\} = \sum_{\mu,\nu=1}^{3} \delta_{\mu\nu}\left(-x_\mu^2 p_\nu^2 + 4 x_\mu x_\nu p_\mu p_\nu\right) = 3\left(\sum_{\mu}^{3} x_\mu^2 p_\mu^2\right) \ . \tag{A.132}$$

Finally, (A.124) in (A.132) yields

$$\{21, 12\} = 3(22) \ . \tag{A.133}$$

This confirms the statement that there will be no constant terms in the commutators. Proceeding in a similar way, one reproduces Table 21 of Section 3.5 and the equations for the constants $\alpha, \ldots, \zeta$ are given by the following matrix:

* Sector α and β_1, β_2:

$$\begin{pmatrix}
-18 & -\frac{5}{2} & 3 & 0 & 0 & 0 & 0 \\
36 & \frac{11}{2} & -9 & 0 & 0 & 0 & 0 \\
-18 & -3 & 6 & 0 & 0 & 0 & 0 \\
18 & \frac{5}{2} & -3 & 0 & 0 & 0 & 0 \\
-36 & -\frac{13}{2} & 15 & 0 & 0 & 0 & 0 \\
18 & 4 & -12 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} & 3 & 0 & 0 & 0 & 0 \\
0 & \frac{3}{2} & -9 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & -3 & 0 & 0 & 0 & 0 \\
0 & -\frac{3}{2} & 9 & 0 & 0 & 0 & 0 \\
 & & \vdots & & & &
\end{pmatrix} \ . \tag{A.134}$$

* Sector β_1, β_2, γ:

$$\begin{pmatrix} & & & \vdots & & & \\ 0 & 3 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -9 & 18 & 1 & 0 & 0 & 0 \\ 0 & 0 & 18 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 12 & -36 & -1 & 0 & 0 & 0 \\ 0 & -15 & 36 & \frac{3}{2} & 0 & 0 & 0 \\ 0 & 15 & 18 & -3 & 0 & 0 & 0 \\ 0 & -6 & -36 & 2 & 0 & 0 & 0 \\ 0 & -6 & 0 & 1 & 0 & 0 & 0 \\ 0 & 12 & -36 & -1 & 0 & 0 & 0 \\ 0 & -6 & -36 & 2 & 0 & 0 & 0 \\ 0 & 6 & -18 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -6 & 72 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -6 & 36 & 0 & 0 & 0 & 0 \\ 0 & 6 & -36 & 0 & 0 & 0 & 0 \\ & & & \vdots & & & \end{pmatrix} . \tag{A.135}$$

* Sector δ, ϵ, ζ:

$$\begin{pmatrix} & & & \vdots & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 12 & \frac{5}{2} & 0 \\ 0 & 0 & 0 & 0 & -12 & -3 & 0 \\ 0 & 0 & 0 & 0 & -12 & -\frac{5}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 12 & 4 & 0 \end{pmatrix} . \tag{A.136}$$

Solving the system of equations with help of Maple™, the solution reads:

$$\alpha = \alpha;\ \beta_1 = -\frac{3}{2}\alpha;\ \beta_2 = -9\alpha;\ \gamma = -54\alpha;\ \delta = 0;\ \epsilon = 0;\ \zeta = 0\ . \tag{A.137}$$

A.6 Composed Weyl-ordered products and Reduction Formulas

In this section the proofs of the different expressions for composed Weyl-Products are given. We start with useful formulae associated to pure products:

$$(a|b) = \frac{1}{2!}\left[ab + ba\right] = (b|a) \ , \tag{A.138}$$

$$(a|b|c) = \frac{1}{3}\left[a\,(b|c) + b\,(a|c) + c\,(a|b)\right] = \frac{1}{3}\left[(a|b)\,c + (b|c)\,a + (c|a)\,b\right] \ , \tag{A.139}$$

$$\begin{aligned}(a|b|c|d) = \frac{1}{4}&\left[a\,(b|c|d) + b\,(a|c|d) + c\,(a|b|d) + d\,(a|b|c)\right]\\ &= \frac{1}{4}\left[(b|c|d)\,a + (c|d|a)\,b + (d|a|b)\,c + (a|b|c)\,d\right]\\ &= \frac{1}{3}\left[((a|b)\,|\,(c|d)) + ((a|c)\,|\,(b|d)) + ((a|d)\,|\,(b|c))\right] \ .\end{aligned} \tag{A.140}$$

A) $(a|(b|c))$. We start with the definition of the Weyl-Product $(a|b|c)$:

$$\begin{aligned}3(a|b|c) &= a(b|c) + b(a|c) + c(a|b)\\ 3(a|b|c) &= a(b|c) + \frac{bac}{2} + \frac{bca}{2} + \frac{cab}{2} + \frac{cba}{2}\\ 3(a|b|c) &= a(b|c) + (b|c)a + \frac{bac + cab}{2} \ .\end{aligned} \tag{A.141}$$

In the same way:

$$2(a|(b|c)) = a(b|c) + (b|c)a \ . \tag{A.142}$$

Subtracting the equations (A.141) and (A.142):

$$\begin{aligned}6\left[(a|(b|c)) - (a|b|c)\right] &= a(b|c) + (b|c)a - (bac + cab)\\ &= \frac{abc}{2} + \frac{acb}{2} + \frac{bca}{2} + \frac{cba}{2} - \frac{bac}{2} - \frac{bac}{2} - \frac{cab}{2} - \frac{cab}{2}\\ &= \frac{1}{2}\left([a,b]c - c[a,b] - b[a,c] + [a,c]b\right)\\ &= \frac{1}{2}\left([[a,b],c] + [[a,c],b]\right) \ .\end{aligned} \tag{A.143}$$

Defining

$$R^a_{bc} = \frac{1}{12}\left([[a,b],c] + [[a,c],b]\right) \ , \tag{A.144}$$

and inserting in (A.143) we arrive to the desired result:

$$\begin{aligned}6\left((a|(b|c)) - (a|b|c)\right) &= \frac{1}{2}\left([[a,b],c] + [[a,c],b]\right)\\ \Longrightarrow (a|(b|c)) &= (a|b|c) + R^a_{bc} \ .\end{aligned} \tag{A.145}$$

B) $(a|b|(c|d))$. We insert this proof because it is very didactical when calculating more complicated composed products. Let us start by defining $\zeta := ((a|b)|(c|d))$ with the help of (A.145):

$$\zeta := ((a|b)|(c|d)) = ((a|b)|c|d) + R_{cd}^{(a|b)} \ , \tag{A.146}$$

$$((a|b)|(c|d)) = ((c|d)|a|b) + R_{ab}^{(c|d)} \ . \tag{A.147}$$

Additionally, we define the following symbol for reducing computations:

$$< x|y|z > := xyz - zyx \ , \tag{A.148}$$

satisfying the properties:

$$< x + y|u|v > = < x|u|v > + < y|u|v > \ , \tag{A.149}$$

$$< x|u|y + z > = < x|u|y > + < x|u|z > \ , \tag{A.150}$$

$$< x|u + v|y > = < x|u|y > + < x|v|y > \ . \tag{A.151}$$

From the equations (A.146) and (A.147), we can conclude

$$((a|b)|c|d) + R_{cd}^{(a|b)} = ((c|d)|a|b) + R_{ab}^{(c|d)} \ , \tag{A.152}$$

$$\Longrightarrow ((a|b)|c|d) - ((c|d)|a|b) = R_{ab}^{(c|d)} - R_{cd}^{(a|b)} \ . \tag{A.153}$$

We calculate the difference on the left-hand side:

$$((a|b)|c|d) - ((c|d)|a|b) = \frac{1}{6}\left\{\begin{array}{c} (a|b)cd \\ (a|b)dc \\ c(a|b)d \\ cd(a|b) \\ d(a|b)c \\ dc(a|b) \end{array}\right\} - \frac{1}{6}\left\{\begin{array}{c} (c|d)ab \\ (c|d)ba \\ a(c|d)b \\ ab(c|d) \\ b(c|d)a \\ ba(c|d) \end{array}\right\} , \tag{A.154}$$

$$((a|b)|c|d) - ((c|d)|a|b) = \frac{1}{6}\left\{\begin{array}{c} 2(a|b)(c|d) \\ < c|(a|b)|d > \\ 2(c|d)(a|b) \end{array}\right\} - \frac{1}{6}\left\{\begin{array}{c} 2(c|d)(a|b) \\ < a|(c|d)|b > \\ 2(a|b)(c|d) \end{array}\right\} , \tag{A.155}$$

$$\therefore ((a|b)|c|d) - ((c|d)|a|b) = \frac{1}{6} < c|(a|b)|d > -\frac{1}{6} < a|(c|d)|b > = R_{ab}^{(c|d)} - R_{cd}^{(a|b)} \ . \tag{A.156}$$

In a similar way:

$$((a|b)|c|d) + ((c|d)|a|b) = \frac{1}{6}\left\{\begin{array}{c} 2(a|b)(c|d) \\ < c|(a|b)|d > \\ 2(c|d)(a|b) \end{array}\right\} + \frac{1}{6}\left\{\begin{array}{c} 2(c|d)(a|b) \\ < a|(c|d)|b > \\ 2(a|b)(c|d) \end{array}\right\} , \tag{A.157}$$

$$\Longrightarrow ((a|b)|c|d) + ((c|d)|a|b) = \frac{8}{6}((a|b)|(c|d)) + \frac{1}{6} < c|(a|b)|d > + < a|(c|d)|b > \ . \quad (A.158)$$

Then, adding the equations (A.146) and (A.147) we get

$$2((a|b)|(c|d)) = \frac{4}{3}((a|b)|(c|d)) + \frac{1}{6} < c|(a|b)|d > + \frac{1}{6} < a|(c|d)|b > + R_{cd}^{(a|b)} + R_{ab}^{(c|d)} \ , \quad (A.159)$$

$$\Longrightarrow \frac{2}{3}((a|b)|(c|d)) - \frac{1}{6} < c|(a|b)|d > - \frac{1}{6} < a|(c|d)|b >= R_{cd}^{(a|b)} + R_{ab}^{(c|d)} \ . \quad (A.160)$$

Recalling the result (A.147)

$$\begin{aligned} (a|b|(c|d)) + R_{ab}^{(c|d)} &= ((a|b)|(c|d)) \\ &= 3(a|b|c|d) - ((a|c)|(b|d)) - ((a|d)|(b|c)) \\ &= (a|b|c|d) + 2(a|b|c|d) - ((a|c)|(b|d)) - ((a|d)|(b|c)) \\ &= (a|b|c|d) + \frac{2}{3}((a|b)|(c|d)) + \\ &\quad -\frac{1}{3}((a|c)|(b|d)) - \frac{1}{3}((a|d)|(b|c)) \ , \end{aligned} \quad (A.161)$$

and solving for $(a|b|(c|d))$, one gets

$$(a|b|(c|d)) = (a|b|c|d) + \frac{2}{3}((a|b)|(c|d)) - R_{ab}^{(c|d)} - \frac{1}{3}((a|c)|(b|d)) - \frac{1}{3}((a|d)|(b|c)) \ . \quad (A.162)$$

Putting in the right-hand side of the last equation the expression for $R_{ab}^{(c|d)}$ that follows from (A.160):

$$\begin{aligned} (a|b|(c|d)) = (a|b|c|d) + R_{cd}^{(a|b)} + \frac{1}{6} < c|(a|b)|d > + \frac{1}{6} < a|(c|d)|b > + \\ - \frac{1}{3} \left[((a|c)|(b|d)) + ((a|d)|(b|c)) \right] \| . \end{aligned} \quad (A.163)$$

There is an extra property, whose proof is straightforward using direct calculation:

$$\begin{aligned} ((a|c)|(b|d)) + ((a|d)|(b|c)) - \frac{1}{2} < c|(a|b)|d > - \frac{1}{2} < a|(c|d)|b >= \\ = \frac{1}{4}([a,c]|[b,d]) + \frac{1}{4}([a,d]|[b,c]) \ , \end{aligned} \quad (A.164)$$

and putting this result in the equation (A.163), the expression reduces considerably:

$$\Longrightarrow (a|b|(c|d)) = (a|b|c|d) + R_{cd}^{(a|b)} - \frac{1}{12}([a,c]|[b,d]) - \frac{1}{12}([a,d]|[b,c]) \ . \quad (A.165)$$

Finally, inserting the definition (A.174) of $R_{cd}^{(a|b)}$

$$R_{cd}^{(a|b)} = (a|R_{cd}^{b}) + (b|R_{cd}^{a}) + \frac{1}{6}([a,c]|[b,d]) + \frac{1}{6}([a,d]|[b,c]) \quad (A.166)$$

in the equation (A.165), we arrive to the desired result:

$$(a|b|(c|d)) = (a|b|c|d) + (a|R_{cd}^{b}) + (b|R_{cd}^{a}) + \frac{1}{12}([a,c]|[b,d]) + \frac{1}{12}([a,d]|[b,c]) \ . \quad (A.167)$$

C) For the rest of the needed composed products, we apply the following idea: more complicated products must reproduce in limiting cases the already known formulae. For example, given the reduction formula

$$(a|b|(d|e)) = (a|b|d|e) + (a|R^b_{de}) + (b|R^a_{de}) + \frac{1}{12}([a,d]|[b,e]) + \frac{1}{12}([a,e]|[b,d]) \ , \quad \text{(A.168)}$$

the next case $(a|b|c|(d|e))$ should contain this previous one when $c \to 1$, so we can postulate:

$$\begin{aligned}(a|b|c|(d|e)) &= (a|b|c|d|e) + (a|b|R^c_{de}) + (a|c|R^b_{de}) + (b|c|R^a_{de}) \\ &+ \frac{1}{12}(a|[b,d]|[c,e]) + \frac{1}{12}(a|[b,e]|[c,d]) + \frac{1}{12}(b|[a,d]|[c,e]) + \frac{1}{12}(b|[a,e]|[c,d]) \\ &+ \frac{1}{12}(c|[a,d]|[b,e]) + \frac{1}{12}(c|[a,e]|[b,d]) + \text{new terms} \ . \end{aligned} \quad \text{(A.169)}$$

Which form should exhibit those new terms? We have two possibilities, trying to form new rests:

$$(a|[b,d]|[c,e]) \longrightarrow R^a_{[b,d],[c,e]} \ , \quad \text{(A.170)}$$

$$(a|b|R^c_{de}) \longrightarrow R^{R^c_{de}}_{ab} \ . \quad \text{(A.171)}$$

So, we reduce the problem to determine constants "x" and "y" for reproducing the full expansion of $(a|b|c|(d|e))$:

$$\begin{aligned}(a|b|c|(d|e)) &= (a|b|c|d|e) + (a|b|R^c_{de}) + (a|c|R^b_{de}) + (b|c|R^a_{de}) + \\ &+ \frac{1}{12}(a|[b,d]|[c,e]) + \frac{1}{12}(a|[b,e]|[c,d]) + \frac{1}{12}(b|[a,d]|[c,e]) + \frac{1}{12}(b|[a,e]|[c,d]) \\ &+ \frac{1}{12}(c|[a,d]|[b,e]) + \frac{1}{12}(c|[a,e]|[b,d]) + \\ &+ x \cdot \begin{bmatrix} R^a_{[b,d][c,e]} + R^a_{[b,e][c,d]} \\ R^b_{[a,d][c,e]} + R^b_{[a,e][c,d]} \\ R^c_{[a,d][b,e]} + R^c_{[a,e][b,d]} \end{bmatrix} + y \cdot \begin{bmatrix} R^{R^a_{de}}_{bc} \\ R^{R^b_{de}}_{ac} \\ R^{R^c_{de}}_{ab} \end{bmatrix} \ . \end{aligned} \quad \text{(A.172)}$$

Please note, we are not working with matrices, it is only notation:

$$y \cdot \begin{bmatrix} R^{R^a_{de}}_{bc} \\ R^{R^b_{de}}_{ac} \\ R^{R^c_{de}}_{ab} \end{bmatrix} := y \cdot \left[R^{R^a_{de}}_{bc} + R^{R^b_{de}}_{ac} + R^{R^c_{de}}_{ab} \right] \ . \quad \text{(A.173)}$$

With the help of Maple™ a simple program was written for expanding those products; identifying similar terms, and solving the respective system of equations, a solution for x and y was found. The same procedure is repeated for the rest of the composed products appearing in the calculations for $[\mathcal{C}, 30]$ and $[\mathcal{C}, 21]$. We just summarize the main formulae obtained with this method:

$$R^{(a|b)}_{cd} = (a|R^b_{cd}) + (b|R^a_{cd}) + \frac{1}{6}([a,c]|[b,d]) + \frac{1}{6}([a,d]|[b,c]) \ , \tag{A.174}$$

$$\begin{aligned} R^{(a|b|c)}_{de} = (a|b|R^c_{de}) + (a|c|R^b_{de}) + (b|c|R^a_{de}) + \\ + \frac{1}{6}(a|[b,d]|[c,e]) + \frac{1}{6}(a|[b,e]|[c,d]) + \\ + \frac{1}{6}(b|[a,d]|[c,e]) + \frac{1}{6}(b|[a,e]|[c,d]) + \\ + \frac{1}{6}(c|[a,d]|[b,e]) + \frac{1}{6}(c|[a,e]|[b,d]) \ , \end{aligned} \tag{A.175}$$

$$(a|(b|c)) = (a|b|c) + R^a_{bc} \ , \tag{A.176}$$

$$(a|(b|c|d)) = (a|b|c|d) + \begin{bmatrix} (b|R^a_{cd}) \\ (c|R^a_{bd}) \\ (d|R^a_{bc}) \end{bmatrix} \ , \tag{A.177}$$

$$(a|(b|c|d|e)) = (a|b|c|d|e) + \begin{bmatrix} (b|c|R^a_{de}) \\ (b|d|R^a_{ce}) \\ (b|e|R^a_{cd}) \\ (c|d|R^a_{be}) \\ (c|e|R^a_{bd}) \\ (d|e|R^a_{bc}) \end{bmatrix} - \left(\frac{1}{5}\right) \begin{bmatrix} R^{R^a_{de}}_{bc} \\ R^{R^a_{ce}}_{bd} \\ R^{R^a_{cd}}_{be} \\ R^{R^a_{be}}_{cd} \\ R^{R^a_{bd}}_{ce} \\ R^{R^a_{bc}}_{de} \end{bmatrix} \ , \tag{A.178}$$

$$
(a|(b|c|d|e|f)) = (a|b|c|d|e|f) + \begin{bmatrix} (b|c|d|R^a_{ef}) \\ (b|c|e|R^a_{df}) \\ (b|d|e|R^a_{cf}) \\ (c|d|e|R^a_{bf}) \\ (c|e|f|R^a_{de}) \\ (b|d|f|R^a_{ce}) \\ (c|d|f|R^a_{be}) \\ (b|e|f|R^a_{cd}) \\ (c|e|f|R^a_{bd}) \\ (d|e|f|R^a_{bc}) \end{bmatrix} +
$$

$$
- \left(\frac{1}{5}\right) \begin{bmatrix} (b|R_{cd}^{R^a_{ef}}) + (c|R_{bd}^{R^a_{ef}}) + (d|R_{bc}^{R^a_{ef}}) \\ (b|R_{ce}^{R^a_{df}}) + (c|R_{be}^{R^a_{df}}) + (e|R_{bc}^{R^a_{df}}) \\ (b|R_{de}^{R^a_{cf}}) + (d|R_{be}^{R^a_{cf}}) + (e|R_{bd}^{R^a_{cf}}) \\ (c|R_{de}^{R^a_{bf}}) + (d|R_{be}^{R^a_{ef}}) + (e|R_{cd}^{R^a_{bf}}) \\ (b|R_{cf}^{R^a_{de}}) + (c|R_{bf}^{R^a_{de}}) + (f|R_{bc}^{R^a_{de}}) \\ (b|R_{df}^{R^a_{ce}}) + (d|R_{bf}^{R^a_{ce}}) + (f|R_{bd}^{R^a_{ce}}) \\ (c|R_{df}^{R^a_{be}}) + (d|R_{cf}^{R^a_{be}}) + (f|R_{cd}^{R^a_{be}}) \\ (b|R_{ef}^{R^a_{cd}}) + (e|R_{bf}^{R^a_{cd}}) + (f|R_{be}^{R^a_{cd}}) \\ (c|R_{ef}^{R^a_{bd}}) + (e|R_{cf}^{R^a_{bd}}) + (f|R_{ce}^{R^a_{bd}}) \\ (d|R_{ef}^{R^a_{bc}}) + (e|R_{df}^{R^a_{bc}}) + (f|R_{de}^{R^a_{bc}}) \end{bmatrix} , \tag{A.179}
$$

$$
(a|b|(c|d)) = (a|b|c|d) + \begin{bmatrix} (a|R^b_{cd}) \\ (b|R^a_{cd}) \end{bmatrix} + \left(\frac{1}{12}\right) \begin{bmatrix} ([a,c]|[b,d]) \\ ([a,d]|[b,c]) \end{bmatrix} , \tag{A.180}
$$

$$
(a|b|c|(d|e)) = (a|b|c|d|e) + \begin{bmatrix} (a|b|R^c_{de}) \\ (a|c|R^b_{de}) \\ (b|c|R^a_{de}) \end{bmatrix} + \left(\frac{1}{12}\right) \begin{bmatrix} (a|[b,d]|[c,e]) + (a|[b,e]|[c,d]) \\ (b|[a,d]|[c,e]) + (b|[a,e]|[c,d]) \\ (c|[a,d]|[b,e]) + (c|[a,e]|[b,d]) \end{bmatrix}
$$

$$
+ \left(\frac{1}{30}\right) \begin{bmatrix} R^a_{[b,d][c,e]} + R^a_{[b,e][c,d]} \\ R^b_{[a,d][c,e]} + R^b_{[a,e][c,d]} \\ R^c_{[a,d][b,e]} + R^c_{[a,e][b,d]} \end{bmatrix} - \left(\frac{1}{5}\right) \begin{bmatrix} R_{bc}^{R^a_{de}} \\ R_{ac}^{R^b_{de}} \\ R_{ab}^{R^c_{de}} \end{bmatrix} , \tag{A.181}
$$

$$
(a|b|c|d|(e|f)) = (a|b|c|d|e|f) + \begin{bmatrix} (a|b|c|R^d_{ef}) \\ (a|b|d|R^c_{ef}) \\ (a|c|d|R^b_{ef}) \\ (b|c|d|R^a_{ef}) \end{bmatrix} + \left(\frac{1}{12}\right) \begin{bmatrix} ([a,e]|[b,f]|c|d) + ([a,f]|[b,e]|c|d) \\ ([a,e]|[c,f]|b|d) + ([a,f]|[c,e]|b|d) \\ ([a,e]|[d,f]|b|c) + (a[,f]|[d,e]|b|c) \\ ([b,e]|[c,f]|a|d) + ([b,f]|[c,e]|a|d) \\ ([b,e]|[d,f]|a|c) + ([b,f]|[d,e]|a|c) \\ ([c,e]|[d,f]|a|b) + ([c,f]|[d,e]|a|b) \end{bmatrix}
$$

$$
- \left(\frac{1}{30}\right) \begin{bmatrix} (R^a_{[b,e][c,f]}|d) + (R^a_{[b,f][c,e]}|d) \\ (R^a_{[b,e][d,f]}|c) + (R^a_{[b,f][d,e]}|c) \\ (R^a_{[c,e][d,f]}|b) + (R^a_{[c,f][d,e]}|b) \\ (R^b_{[a,e][c,f]}|d) + (R^b_{[a,f][c,e]}|d) \\ (R^b_{[a,e][d,f]}|c) + (R^b_{[a,f][d,e]}|c) \\ (R^b_{[c,e][d,f]}|a) + (R^b_{[c,f][d,e]}|a) \\ (R^c_{[a,e][b,f]}|d) + (R^c_{[a,f][b,e]}|d) \\ (R^c_{[b,e][d,f]}|a) + (R^c_{[b,f][d,e]}|a) \\ (R^c_{[a,e][d,f]}|b) + (R^c_{[a,f][d,e]}|b) \\ (R^d_{[b,e][c,f]}|a) + (R^d_{[b,f][c,e]}|a) \\ (R^d_{[a,e][b,f]}|c) + (R^d_{[a,f][b,e]}|c) \\ (R^d_{[a,e][c,f]}|b) + (R^d_{[a,f][c,e]}|b) \end{bmatrix} - \left(\frac{1}{5}\right) \begin{bmatrix} (R_{bc}^{R^d_{ef}}|a) + (R_{ac}^{R^d_{ef}}|b) + (R_{ab}^{R^d_{ef}}|c) \\ (R_{bd}^{R^c_{ef}}|a) + (R_{ad}^{R^c_{ef}}|b) + (R_{ab}^{R^c_{ef}}|d) \\ (R_{cd}^{R^b_{ef}}|a) + (R_{ac}^{R^b_{ef}}|d) + (R_{ad}^{R^b_{ef}}|c) \\ (R_{cd}^{R^a_{ef}}|b) + (R_{bc}^{R^a_{ef}}|d) + (R_{bd}^{R^a_{ef}}|c) \end{bmatrix}
$$

$$
+ \left(\frac{1}{720}\right) \begin{bmatrix} ([[a,e],b]|[[c,f],d]) + ([[a,e],b]|[[d,f],c]) \\ ([[a,e],c]|[[b,f],d]) + ([[a,e],c]|[[d,f],b]) \\ ([[a,e],d]|[[b,f],c]) + ([[a,e],d]|[[c,f],b]) \\ ([[a,f],b]|[[c,e],d]) + ([[a,f],b]|[[d,e],c]) \\ ([[a,f],c]|[[b,e],d]) + ([[a,f],c]|[[d,e],b]) \\ ([[a,f],d]|[[b,e],c]) + ([[a,f],d]|[[c,e],b]) \\ ([[b,e],a]|[[c,f],d]) + ([[b,e],a]|[[d,f],c]) \\ ([[b,e],c]|[[d,f],a]) + ([[b,e],d]|[[c,f],a]) \\ ([[b,f],a]|[[c,e],d]) + ([[b,f],a]|[[d,e],c]) \\ ([[b,f],c]|[[d,e],a]) + ([[b,f],d]|[[c,e],a]) \\ ([[c,e],a]|[[d,f],b]) + ([[c,e],b]|[[d,f],a]) \\ ([[c,f],a]|[[d,e],b]) + ([[c,f],b]|[[d,e],a]) \end{bmatrix} - \left(\frac{1}{60}\right) \begin{bmatrix} \left([a,e]|R_{cd}^{[b,f]}\right) + \left([a,f]|R_{cd}^{[b,e]}\right) \\ \left([a,e]|R_{bd}^{[c,f]}\right) + \left([a,f]|R_{bd}^{[c,e]}\right) \\ \left([a,e]|R_{bc}^{[d,f]}\right) + \left([a,f]|R_{bc}^{[d,e]}\right) \\ \left([b,e]|R_{cd}^{[a,f]}\right) + \left([b,f]|R_{cd}^{[a,e]}\right) \\ \left([b,e]|R_{ad}^{[c,f]}\right) + \left([b,f]|R_{ad}^{[c,e]}\right) \\ \left([b,e]|R_{ac}^{[d,f]}\right) + \left([b,f]|R_{ac}^{[d,e]}\right) \\ \left([c,e]|R_{bd}^{[a,f]}\right) + \left([c,f]|R_{bd}^{[a,e]}\right) \\ \left([c,e]|R_{ad}^{[b,f]}\right) + \left([c,f]|R_{ad}^{[b,e]}\right) \\ \left([c,e]|R_{ab}^{[d,f]}\right) + \left([c,f]|R_{ab}^{[d,e]}\right) \\ \left([d,e]|R_{bc}^{[a,f]}\right) + \left([d,f]|R_{bc}^{[a,e]}\right) \\ \left([d,e]|R_{ac}^{[b,f]}\right) + \left([d,f]|R_{ac}^{[b,e]}\right) \\ \left([d,e]|R_{ab}^{[c,f]}\right) + \left([d,f]|R_{ab}^{[c,e]}\right) \end{bmatrix}. \tag{A.182}
$$

A.7 Contributions from composed Weyl-ordered products

The following tables show the results when expanding non-pure Weyl-ordered products appearing in $[\mathcal{C}, B_{30}]$ and $[\mathcal{C}, B_{21}]$, using the formulae (A.174) to (A.182) of section A.6.

$\frac{1}{6}((20\vert 20)\vert 30\vert 11\vert 02\vert 02)$	$\frac{1}{6}((20\vert 11)\vert 30\vert 20\vert 02\vert 02)$	$-\frac{1}{6}((20\vert 02)\vert 30\vert 20\vert 11\vert 02)$	$\frac{1}{3}((11\vert 11)\vert 30\vert 20\vert 11\vert 02)$
$\frac{1}{6}(30\vert 20\vert 20\vert 11\vert 02\vert 02)$ $-\frac{8}{9}(30\vert 20\vert 11\vert 02)$ $-\frac{4}{9}(30\vert 11\vert 11\vert 11)$ $+\frac{74}{135}(30\vert 11)$ $+\frac{2}{9}(21\vert 20)$	$\frac{1}{6}(30\vert 20\vert 20\vert 11\vert 02\vert 02)$ $+\frac{1}{9}(30\vert 20\vert 11\vert 02)$ $+\frac{2}{9}(21\vert 20)$ $-\frac{19}{135}(30\vert 11)$	$-\frac{1}{6}(30\vert 20\vert 20\vert 11\vert 02\vert 02)$ $-\frac{5}{6}(30\vert 20\vert 11\vert 02)$ $-\frac{1}{6}(21\vert 20\vert 20\vert 02)$ $-\frac{1}{3}(21\vert 20\vert 11\vert 11)$ $-\frac{2}{9}(30\vert 11\vert 11\vert 11)$ $+\frac{37}{90}(30\vert 11)$ $+\frac{37}{90}(21\vert 20)$	$\frac{1}{3}(30\vert 20\vert 11\vert 11\vert 11\vert 02)$ $-\frac{13}{18}(30\vert 20\vert 11\vert 02)$ $+\frac{199}{270}(30\vert 11)$ $+\frac{1}{9}(21\vert 20)$
$\frac{1}{6}((20\vert 11)\vert 30\vert 11\vert 11\vert 02)$	$-\frac{1}{6}((20\vert 02)\vert 30\vert 11\vert 11\vert 11)$	$\frac{1}{3}((11\vert 11)\vert 30\vert 11\vert 11\vert 11)$	$\frac{1}{6}((20\vert 20)\vert 21\vert 20\vert 02\vert 02)$
$\frac{1}{6}(30\vert 20\vert 11\vert 11\vert 11\vert 02)$ $+\frac{1}{18}(30\vert 11\vert 11\vert 11)$ $+\frac{1}{6}(30\vert 20\vert 11\vert 02)$ $-\frac{13}{180}(30\vert 11)$	$-\frac{1}{6}(30\vert 20\vert 11\vert 11\vert 11\vert 02)$ $-\frac{5}{6}(30\vert 11\vert 11\vert 11)$ $-\frac{1}{2}(21\vert 20\vert 11\vert 11)$ $-\frac{1}{3}(30\vert 20\vert 11\vert 02)$ $-\frac{59}{180}(30\vert 11)$ $-\frac{3}{20}(21\vert 20)$	$\frac{1}{3}(30\vert 11\vert 11\vert 11\vert 11\vert 11)$ $-\frac{1}{2}(30\vert 11\vert 11\vert 11)$ $-\frac{9}{20}(30\vert 11)$	$\frac{1}{6}(21\vert 20\vert 20\vert 20\vert 02\vert 02)$ $-\frac{4}{9}(21\vert 20\vert 20\vert 02)$ $-\frac{4}{9}(30\vert 20\vert 11\vert 02)$ $-\frac{4}{9}(21\vert 20\vert 11\vert 11)$ $+\frac{8}{45}(21\vert 20)$ $+\frac{32}{135}(30\vert 11)$
$\frac{1}{6}((20\vert 20)\vert 21\vert 11\vert 11\vert 02)$	$\frac{1}{6}((20\vert 20)\vert 20\vert 20\vert 03\vert 02)$	$-\frac{1}{6}((20\vert 02)\vert 21\vert 20\vert 20\vert 02)$	$\frac{1}{3}((11\vert 11)\vert 21\vert 20\vert 20\vert 02)$
$\frac{1}{6}(21\vert 20\vert 20\vert 11\vert 11\vert 02)$ $-\frac{2}{3}(21\vert 20\vert 11\vert 11)$ $-\frac{2}{9}(30\vert 20\vert 11\vert 02)$ $-\frac{2}{9}(30\vert 11\vert 11\vert 11)$ $-\frac{1}{9}(21\vert 20\vert 20\vert 02)$ $-\frac{11}{45}(30\vert 11)$ $-\frac{1}{9}(21\vert 20)$	$\frac{1}{6}(20\vert 20\vert 20\vert 20\vert 03\vert 02)$ $-\frac{2}{9}(20\vert 20\vert 20\vert 03)$ $-\frac{2}{3}(21\vert 20\vert 20\vert 02)$ $-\frac{2}{3}(20\vert 20\vert 12\vert 11)$ $+\frac{68}{45}(21\vert 20)$ $-\frac{8}{45}(30\vert 11)$	$-\frac{1}{6}(21\vert 20\vert 20\vert 20\vert 02\vert 02)$ $-\frac{13}{18}(21\vert 20\vert 20\vert 02)$ $-\frac{2}{9}(30\vert 20\vert 11\vert 02)$ $-\frac{2}{9}(20\vert 20\vert 12\vert 11)$ $-\frac{4}{9}(21\vert 20\vert 11\vert 11)$ $+\frac{91}{135}(21\vert 20)$ $+\frac{2}{27}(30\vert 11)$	$\frac{1}{3}(21\vert 20\vert 20\vert 11\vert 11\vert 02)$ $-\frac{11}{18}(21\vert 20\vert 20\vert 02)$ $+\frac{113}{135}(21\vert 20)$ $+\frac{8}{135}(30\vert 11)$
$\frac{1}{6}((20\vert 20)\vert 20\vert 12\vert 11\vert 02)$	$\frac{1}{6}((20\vert 11)\vert 21\vert 20\vert 11\vert 02)$	$\frac{1}{6}((20\vert 20)\vert 20\vert 11\vert 11\vert 03)$	$-\frac{1}{6}((20\vert 02)\vert 21\vert 20\vert 11\vert 11)$
$\frac{1}{6}(20\vert 20\vert 20\vert 12\vert 11\vert 02)$ $-\frac{4}{9}(20\vert 20\vert 12\vert 11)$ $-\frac{2}{9}(30\vert 20\vert 11\vert 02)$ $-\frac{2}{9}(21\vert 20\vert 20\vert 02)$ $-\frac{4}{9}(21\vert 20\vert 11\vert 11)$ $+\frac{38}{135}(30\vert 11)$ $+\frac{22}{135}(21\vert 20)$	$\frac{1}{6}(21\vert 20\vert 20\vert 11\vert 11\vert 02)$ $+\frac{1}{18}(21\vert 20\vert 11\vert 11)$ $+\frac{1}{12}(21\vert 20\vert 20\vert 02)$ $+\frac{1}{9}(30\vert 20\vert 11\vert 02)$ $-\frac{14}{135}(30\vert 11)$ $-\frac{59}{540}(21\vert 20)$	$\frac{1}{6}(20\vert 20\vert 20\vert 11\vert 11\vert 03)$ $-\frac{2}{3}(21\vert 20\vert 11\vert 11)$ $-\frac{1}{9}(20\vert 20\vert 20\vert 03)$ $-\frac{2}{3}(20\vert 20\vert 12\vert 11)$ $-\frac{1}{9}(21\vert 20)$	$-\frac{1}{6}(21\vert 20\vert 20\vert 11\vert 11\vert 02)$ $-\frac{7}{6}(21\vert 20\vert 11\vert 11)$ $-\frac{1}{9}(30\vert 11\vert 11\vert 11)$ $-\frac{1}{9}(30\vert 20\vert 11\vert 02)$ $-\frac{2}{9}(20\vert 20\vert 12\vert 11)$ $-\frac{1}{9}(21\vert 20\vert 20\vert 02)$ $-\frac{73}{270}(30\vert 11)$ $-\frac{229}{540}(21\vert 20)$

Table 23: Composed Weyl-ordered products from $[\mathcal{C}_\beta, 30] = 0$ (part 1).

$\frac{1}{3}((11\|11)\|21\|20\|11\|11)$	$\frac{1}{6}((20\|20)\|12\|11\|11\|11)$	$\frac{1}{6}((20\|11)\|21\|11\|11\|11)$	$-\frac{1}{6}((20\|02)\|20\|20\|20\|03)$
$\frac{1}{3}(21\|20\|11\|11\|11\|11)$ $-\frac{7}{18}(21\|20\|11\|11)$ $-\frac{2}{27}(30\|11)$ $-\frac{23}{540}(21\|20)$	$\frac{1}{6}(20\|20\|12\|11\|11\|11)$ $-\frac{2}{9}(30\|11\|11\|11)$ $-\frac{2}{3}(21\|20\|11\|11)$ $-\frac{1}{3}(20\|20\|12\|11)$ $-\frac{1}{9}(30\|11)$ $-\frac{1}{9}(21\|20)$	$\frac{1}{6}(21\|20\|11\|11\|11\|11)$ $+\frac{1}{4}(21\|20\|11\|11)$ $+\frac{1}{9}(30\|11\|11\|11)$ $+\frac{1}{6}(30\|11)$ $+\frac{11}{360}(21\|20)$	$-\frac{1}{6}(20\|20\|20\|20\|03\|02)$ $-\frac{1}{2}(20\|20\|20\|03)$ $-(20\|20\|12\|11)$ $+\frac{14}{15}(21\|20)$ $-\frac{4}{15}(30\|11)$
$\frac{1}{3}((11\|11)\|20\|20\|20\|03)$	$\frac{1}{6}((20\|11)\|20\|20\|12\|02)$	$\frac{1}{6}((20\|11)\|20\|20\|11\|03)$	$-\frac{1}{6}((20\|02)\|20\|20\|12\|11)$
$\frac{1}{3}(20\|20\|20\|11\|11\|03)$ $-\frac{5}{6}(20\|20\|20\|03)$ $+\frac{10}{3}(21\|20)$	$\frac{1}{6}(20\|20\|20\|12\|11\|02)$ $+\frac{1}{18}(20\|20\|12\|11)$ $+\frac{1}{9}(21\|20\|20\|02)$ $-\frac{22}{135}(21\|20)$ $+\frac{2}{135}(30\|11)$	$\frac{1}{6}(20\|20\|20\|11\|11\|03)$ $+\frac{1}{12}(20\|20\|20\|03)$ $-\frac{1}{3}(21\|20)$	$-\frac{1}{6}(20\|20\|20\|12\|11\|02)$ $-\frac{19}{18}(20\|20\|12\|11)$ $-\frac{4}{9}(21\|20\|11\|11)$ $-\frac{1}{9}(21\|20\|20\|02)$ $-\frac{1}{18}(20\|20\|20\|03)$ $+\frac{26}{135}(30\|11)$ $-\frac{2}{45}(21\|20)$
$\frac{1}{3}((11\|11)\|20\|20\|12\|11)$	$\frac{1}{6}((20\|11)\|20\|12\|11\|11)$		
$\frac{1}{3}(20\|20\|12\|11\|11\|11)$ $-\frac{1}{2}(20\|20\|12\|11)$ $+\frac{2}{9}(30\|11)$	$\frac{1}{6}(20\|20\|12\|11\|11\|11)$ $+\frac{1}{6}(20\|20\|12\|11)$ $+\frac{1}{9}(21\|20\|11\|11)$ $-\frac{2}{27}(30\|11)$ $+\frac{1}{54}(21\|20)$		

Table 24: Composed Weyl-ordered products from $[\mathcal{C}_\beta, 30] = 0$ (part 2).

<table>
<tr><td>$-\frac{1}{6}((20|02)|30|30|03)$</td><td>$\frac{1}{3}((11|11)|30|30|03)$</td><td>$\frac{1}{6}((20|20)|30|12|03)$</td><td>$\frac{1}{6}((20|11)|30|21|03)$</td></tr>
<tr><td>$-\frac{1}{6}(30|30|20|03|02)$
$-\frac{1}{2}(30|30|03)$
$-(30|21|12)$
$+\frac{1}{10}(21|20)$
$+\frac{1}{15}(30|11)$</td><td>$\frac{1}{3}(30|30|11|11|03)$
$-(30|30|03)$
$-\frac{3}{5}(21|20)$
$+\frac{6}{5}(30|11)$</td><td>$\frac{1}{6}(30|20|20|12|03)$
$-\frac{2}{9}(30|30|03)$
$-\frac{4}{3}(30|21|12)$
$+\frac{2}{15}(21|20)$
$+\frac{4}{45}(30|11)$</td><td>$\frac{1}{6}(30|21|20|11|03)$
$+\frac{1}{9}(30|30|03)$
$+\frac{1}{8}(21|20)$
$-\frac{11}{60}(30|11)$</td></tr>
<tr><td>$-\frac{1}{6}((20|02)|30|21|12)$</td><td>$\frac{1}{3}((11|11)|30|21|12)$</td><td>$\frac{1}{6}((20|11)|30|12|12)$</td><td>$\frac{1}{6}((20|20)|21|21|03)$</td></tr>
<tr><td>$-\frac{1}{6}(30|21|20|12|02)$
$-\frac{4}{3}(30|21|12)$
$-\frac{1}{18}(30|30|03)$
$-\frac{1}{3}(21|21|21)$
$+\frac{1}{12}(21|20)$
$-\frac{1}{30}(30|11)$</td><td>$\frac{1}{3}(30|21|12|11|11)$
$-\frac{5}{9}(30|21|12)$
$+\frac{1}{18}(21|20)$
$-\frac{7}{135}(30|11)$</td><td>$\frac{1}{6}(30|20|12|12|11)$
$+\frac{2}{9}(30|21|12)$
$-\frac{1}{18}(21|20)$
$+\frac{7}{135}(30|11)$</td><td>$\frac{1}{6}(21|21|20|20|03)$
$-\frac{2}{3}(21|21|21)$
$-\frac{1}{9}(30|30|03)$
$-\frac{2}{3}(30|21|12)$
$-\frac{4}{15}(21|20)$
$+\frac{8}{45}(30|11)$</td></tr>
<tr><td>$-\frac{1}{6}((20|02)|21|21|21)$</td><td>$\frac{1}{3}((11|11)|21|21|21)$</td><td>$\frac{1}{6}((20|20)|21|12|12)$</td><td>$\frac{1}{6}((20|11)|21|21|12)$</td></tr>
<tr><td>$-\frac{1}{6}(21|21|21|20|02)$
$-\frac{7}{6}(21|21|21)$
$-\frac{2}{3}(30|21|12)$
$-\frac{1}{5}(21|20)$
$+\frac{2}{45}(30|11)$</td><td>$\frac{1}{3}(21|21|21|11|11)$
$-\frac{1}{3}(21|21|21)$</td><td>$\frac{1}{6}(21|20|20|12|12)$
$-\frac{8}{9}(30|21|12)$
$-\frac{4}{9}(21|21|21)$
$+\frac{1}{15}(21|20)$
$-\frac{2}{27}(30|11)$</td><td>$\frac{1}{6}(21|21|20|12|11)$
$+\frac{2}{9}(30|21|12)$
$+\frac{1}{9}(21|21|21)$
$-\frac{1}{60}(21|20)$
$+\frac{1}{54}(30|11)$</td></tr>
</table>

Table 25: Composed Weyl-ordered products from $[\mathcal{C}_\gamma, 30] = 0$.

<table>
<tr><td>$\frac{1}{6}((20|11)|30|02)$</td><td>$-\frac{1}{6}((20|02)|30|11)$</td><td>$\frac{1}{3}((11|11)|30|11)$</td><td>$\frac{1}{6}((20|20)|21|02)$</td><td>$\frac{1}{6}((20|20)|20|03)$</td></tr>
<tr><td>$\frac{1}{6}(30|20|11|02)$
$+\frac{1}{18}(30|11)$</td><td>$-\frac{1}{6}(30|20|11|02)$
$-\frac{7}{18}(30|11)$
$-\frac{1}{6}(21|20)$</td><td>$\frac{1}{3}(30|11|11|11)$
$-\frac{1}{2}(30|11)$</td><td>$\frac{1}{6}(21|20|20|02)$
$-\frac{2}{9}(21|20)$
$-\frac{2}{9}(30|11)$</td><td>$\frac{1}{6}(20|20|20|03)$
$-\frac{2}{3}(21|20)$</td></tr>
<tr><td>$-\frac{1}{6}((20|02)|21|20)$</td><td>$\frac{1}{3}((11|11)|21|20)$</td><td>$\frac{1}{6}((20|20)|12|11)$</td><td>$\frac{1}{6}((20|11)|21|11)$</td><td>$\frac{1}{6}((20|11)|20|12)$</td></tr>
<tr><td>$-\frac{1}{6}(21|20|20|02)$
$-\frac{1}{2}(21|20)$
$-\frac{1}{9}(30|11)$</td><td>$\frac{1}{3}(21|20|11|11)$
$-\frac{7}{18}(21|20)$</td><td>$\frac{1}{6}(20|20|12|11)$
$-\frac{2}{9}(30|11)$
$-\frac{2}{9}(21|20)$</td><td>$\frac{1}{6}(21|20|11|11)$
$+\frac{1}{9}(30|11)$
$+\frac{1}{12}(21|20)$</td><td>$\frac{1}{6}(20|20|12|11)$
$+\frac{1}{9}(21|20)$</td></tr>
</table>

Table 26: Composed Weyl-ordered products from $[\mathcal{C}_\epsilon, 30] = 0$.

<table>
<tr><td>$\frac{1}{6}((20|20)|30|02|02|02)$</td><td>$\frac{1}{6}((20|20)|21|11|02|02)$</td><td>$\frac{1}{6}((20|20)|20|12|02|02)$</td><td>$\frac{5}{6}((20|02)|30|20|02|02)$</td></tr>
<tr><td>$\frac{1}{6}(30|20|20|02|02|02)$
$-\frac{2}{3}(30|20|02|02)$
$-\frac{4}{3}(30|11|11|02)$
$+\frac{44}{45}(30|02)$
$+\frac{4}{3}(21|11)$</td><td>$\frac{1}{6}(21|20|20|11|02|02)$
$-\frac{8}{9}(21|20|11|02)$
$-\frac{1}{9}(30|20|02|02)$
$-\frac{4}{9}(30|11|11|02)$
$-\frac{4}{9}(21|11|11|11)$
$-\frac{2}{27}(30|02)$
$-\frac{2}{45}(21|11)$</td><td>$\frac{1}{6}(20|20|20|12|02|02)$
$-\frac{4}{9}(20|20|12|02)$
$-\frac{2}{9}(30|20|02|02)$
$-\frac{8}{9}(21|20|11|02)$
$-\frac{4}{9}(20|12|11|11)$
$+\frac{52}{135}(30|02)$
$+\frac{4}{45}(20|12)$
$+\frac{112}{135}(21|11)$</td><td>$\frac{5}{6}(30|20|20|02|02|02)$
$+\frac{5}{2}(30|20|02|02)$
$+\frac{10}{3}(21|20|11|02)$
$+\frac{20}{9}(30|11|11|02)$
$-\frac{67}{27}(30|02)$
$-\frac{2}{3}(20|12)$
$-\frac{14}{3}(21|11)$</td></tr>
<tr><td>$-\frac{1}{3}((11|11)|30|20|02|02)$</td><td>$\frac{1}{6}((20|20)|20|11|03|02)$</td><td>$\frac{1}{6}((11|02)|30|20|11|02)$</td><td>$\frac{1}{6}((20|20)|12|11|11|02)$</td></tr>
<tr><td>$-\frac{1}{3}(30|20|11|11|02|02)$
$+\frac{11}{18}(30|20|02|02)$
$-\frac{101}{135}(30|02)$
$-\frac{2}{9}(20|12)$
$-\frac{8}{9}(21|11)$</td><td>$\frac{1}{6}(20|20|20|11|03|02)$
$-\frac{4}{9}(20|20|11|03)$
$-\frac{2}{3}(21|20|11|02)$
$-\frac{1}{3}(20|20|12|02)$
$-\frac{2}{3}(20|12|11|11)$
$+\frac{2}{3}(21|11)$
$+\frac{5}{9}(20|12)$</td><td>$\frac{1}{6}(30|20|11|11|02|02)$
$-\frac{1}{36}(30|20|02|02)$
$-\frac{1}{6}(30|11|11|02)$
$-\frac{1}{3}(21|20|11|02)$
$-\frac{7}{540}(30|02)$
$+\frac{4}{45}(21|11)$
$-\frac{1}{15}(20|12)$</td><td>$\frac{1}{6}(20|20|12|11|11|02)$
$-\frac{2}{3}(20|12|11|11)$
$-\frac{2}{9}(30|11|11|02)$
$-\frac{4}{9}(21|20|11|02)$
$-\frac{4}{9}(21|11|11|11)$
$-\frac{1}{9}(20|20|12|02)$
$-\frac{1}{27}(30|02)$
$-\frac{2}{27}(21|11)$
$-\frac{1}{45}(20|12)$</td></tr>
<tr><td>$\frac{5}{6}((20|02)|30|11|11|02)$</td><td>$-\frac{1}{3}((11|11)|30|11|11|02)$</td><td>$\frac{1}{6}((20|20)|11|11|11|03)$</td><td>$\frac{1}{6}((11|02)|30|11|11|11)$</td></tr>
<tr><td>$\frac{5}{6}(30|20|11|11|02|02)$
$+\frac{85}{18}(30|11|11|02)$
$+\frac{5}{3}(21|20|11|02)$
$+\frac{5}{3}(21|11|11|11)$
$+\frac{5}{9}(30|20|02|02)$
$+\frac{1}{12}(30|02)$
$-\frac{31}{18}(21|11)$
$-\frac{4}{9}(20|12)$</td><td>$-\frac{1}{3}(30|11|11|11|11|02)$
$+\frac{7}{18}(30|11|11|02)$
$-\frac{2}{9}(21|11)$
$-\frac{17}{540}(30|02)$</td><td>$\frac{1}{6}(20|20|11|11|11|03)$
$-\frac{2}{3}(21|11|11|11)$
$-\frac{1}{3}(20|20|11|03)$
$-(11|11|20|12)$
$-\frac{1}{3}(21|11)$
$-\frac{1}{6}(20|12)$</td><td>$\frac{1}{6}(30|11|11|11|11|02)$
$-\frac{1}{12}(30|11|11|02)$
$-\frac{1}{3}(21|11|11|11)$
$-\frac{1}{2}(21|11)$
$-\frac{49}{360}(30|02)$</td></tr>
</table>

Table 27: Composed Weyl-ordered products from $[\mathcal{C}_\beta, 21] = 0$ (part 1).

$\frac{1}{6}((11\vert 02)\vert 21\vert 20\vert 20\vert 02)$	$\frac{5}{6}((20\vert 02)\vert 21\vert 20\vert 11\vert 02)$	$-\frac{1}{3}((11\vert 11)\vert 21\vert 20\vert 11\vert 02)$	$\frac{1}{6}((11\vert 02)\vert 21\vert 20\vert 11\vert 11)$
$\frac{1}{6}(21\vert 20\vert 20\vert 11\vert 02\vert 02)$ $-\frac{1}{3}(21\vert 20\vert 11\vert 02)$ $-\frac{1}{9}(20\vert 20\vert 12\vert 02)$ $-\frac{2}{45}(30\vert 02)$ $+\frac{22}{135}(20\vert 12)$ $+\frac{1}{27}(21\vert 11)$	$\frac{5}{6}(21\vert 20\vert 20\vert 11\vert 02\vert 02)$ $+\frac{95}{18}(21\vert 20\vert 11\vert 02)$ $+\frac{5}{9}(30\vert 11\vert 11\vert 02)$ $+\frac{5}{18}(30\vert 20\vert 02\vert 02)$ $+\frac{5}{9}(20\vert 20\vert 12\vert 02)$ $+\frac{10}{9}(20\vert 12\vert 11\vert 11)$ $+\frac{10}{9}(21\vert 11\vert 11\vert 11)$ $+\frac{7}{54}(30\vert 02)$ $-\frac{55}{54}(21\vert 11)$ $-\frac{17}{27}(20\vert 12)$	$-\frac{1}{3}(21\vert 20\vert 11\vert 11\vert 11\vert 02)$ $+\frac{5}{18}(21\vert 20\vert 11\vert 02)$ $-\frac{67}{270}(21\vert 11)$ $+\frac{1}{135}(30\vert 02)$ $-\frac{4}{45}(20\vert 12)$	$\frac{1}{6}(21\vert 20\vert 11\vert 11\vert 11\vert 02)$ $-\frac{1}{18}(21\vert 20\vert 11\vert 02)$ $-\frac{1}{6}(21\vert 11\vert 11\vert 11)$ $-\frac{1}{9}(20\vert 12\vert 11\vert 11)$ $-\frac{107}{540}(21\vert 11)$ $-\frac{17}{270}(20\vert 12)$ $-\frac{2}{135}(30\vert 02)$
$\frac{5}{6}((20\vert 02)\vert 21\vert 11\vert 11\vert 11)$	$-\frac{1}{3}((11\vert 11)\vert 21\vert 11\vert 11\vert 11)$	$\frac{1}{6}((11\vert 02)\vert 20\vert 20\vert 20\vert 03)$	$\frac{5}{6}((20\vert 02)\vert 20\vert 20\vert 12\vert 02)$
$\frac{5}{6}(21\vert 20\vert 11\vert 11\vert 11\vert 02)$ $\frac{95}{18}(21\vert 11\vert 11\vert 11)$ $+\frac{5}{6}(30\vert 11\vert 11\vert 02)$ $+\frac{5}{3}(20\vert 12\vert 11\vert 11)$ $+\frac{5}{3}(21\vert 20\vert 11\vert 02)$ $+\frac{199}{36}(21\vert 11)$ $+\frac{11}{12}(30\vert 02)$ $+\frac{17}{18}(20\vert 12)$	$-\frac{1}{3}(21\vert 11\vert 11\vert 11\vert 11\vert 11)$ $+\frac{1}{18}(21\vert 11\vert 11\vert 11)$ $+\frac{1}{180}(21\vert 11)$	$\frac{1}{6}(20\vert 20\vert 20\vert 11\vert 03\vert 02)$ $-\frac{1}{2}(20\vert 20\vert 11\vert 03)$ $+\frac{2}{3}(20\vert 12)$ $+\frac{2}{3}(21\vert 11)$	$\frac{5}{6}(20\vert 20\vert 20\vert 12\vert 02\vert 02)$ $+\frac{65}{18}(20\vert 20\vert 12\vert 02)$ $+\frac{20}{9}(21\vert 20\vert 11\vert 02)$ $+\frac{20}{9}(20\vert 12\vert 11\vert 11)$ $+\frac{5}{9}(20\vert 20\vert 11\vert 03)$ $-\frac{2}{9}(30\vert 02)$ $-\frac{79}{27}(20\vert 12)$ $-\frac{56}{27}(21\vert 11)$
$-\frac{1}{3}((11\vert 11)\vert 20\vert 20\vert 12\vert 02)$	$\frac{5}{6}((20\vert 02)\vert 20\vert 20\vert 11\vert 03)$	$-\frac{1}{3}((11\vert 11)\vert 20\vert 20\vert 11\vert 03)$	$\frac{1}{6}((11\vert 02)\vert 20\vert 20\vert 12\vert 11)$
$-\frac{1}{3}(20\vert 20\vert 12\vert 11\vert 11\vert 02)$ $+\frac{7}{18}(20\vert 20\vert 12\vert 02)$ $-\frac{2}{15}(30\vert 02)$ $-\frac{1}{3}(20\vert 12)$ $-\frac{16}{45}(21\vert 11)$	$\frac{5}{6}(20\vert 20\vert 20\vert 11\vert 03\vert 02)$ $+\frac{25}{6}(20\vert 20\vert 11\vert 03)$ $+\frac{10}{3}(20\vert 12\vert 11\vert 11)$ $+\frac{5}{6}(20\vert 20\vert 12\vert 02)$ $-2(21\vert 11)$ $-\frac{8}{3}(20\vert 12)$ $+\frac{2}{9}(30\vert 02)$	$-\frac{1}{3}(20\vert 20\vert 11\vert 11\vert 11\vert 03)$ $+\frac{1}{2}(20\vert 20\vert 11\vert 03)$ $-\frac{2}{3}(21\vert 11)$ $-\frac{2}{3}(20\vert 12)$	$\frac{1}{6}(20\vert 20\vert 12\vert 11\vert 11\vert 02)$ $-\frac{1}{36}(20\vert 20\vert 12\vert 02)$ $-\frac{1}{3}(20\vert 12\vert 11\vert 11)$ $-\frac{4}{45}(21\vert 11)$ $-\frac{11}{45}(20\vert 12)$ $+\frac{1}{45}(30\vert 02)$
$\frac{5}{6}((20\vert 02)\vert 20\vert 12\vert 11\vert 11)$	$-\frac{1}{3}((11\vert 11)\vert 20\vert 12\vert 11\vert 11)$		
$\frac{5}{6}(20\vert 20\vert 12\vert 11\vert 11\vert 02)$ $+\frac{35}{6}(20\vert 12\vert 11\vert 11)$ $+\frac{10}{9}(21\vert 11\vert 11\vert 11)$ $+\frac{10}{9}(21\vert 20\vert 11\vert 02)$ $+\frac{5}{9}(20\vert 20\vert 11\vert 03)$ $+\frac{5}{9}(20\vert 20\vert 12\vert 02)$ $+\frac{17}{27}(21\vert 11)$ $+\frac{229}{108}(20\vert 12)$ $-\frac{8}{27}(30\vert 02)$	$-\frac{1}{3}(20\vert 12\vert 11\vert 11\vert 11\vert 11)$ $+\frac{1}{6}(20\vert 12\vert 11\vert 11)$ $+\frac{1}{180}(20\vert 12)$		

Table 28: Composed Weyl-ordered products from $[\mathcal{C}_\beta, 21] = 0$ (part 2).

<table>
<tr><th>$\frac{1}{6}((20|20)|30|03|03)$</th><th>$\frac{1}{6}((11|02)|30|30|03)$</th><th>$\frac{1}{6}((20|20)|21|12|03)$</th><th>$\frac{5}{6}((20|02)|30|21|03)$</th></tr>
<tr><td>$\frac{1}{6}(30|20|20|03|03)$
$-\frac{4}{3}(30|21|03)$
$-(30|12|12)$
$+\frac{1}{5}(20|12)$
$+\frac{2}{5}(21|11)$
$+\frac{1}{15}(30|02)$</td><td>$\frac{1}{6}(30|30|11|03|02)$
$-\frac{2}{3}(30|21|03)$
$-\frac{7}{30}(20|12)$
$+\frac{11}{60}(30|02)$
$+\frac{11}{30}(21|11)$</td><td>$\frac{1}{6}(21|20|20|12|03)$
$-\frac{4}{9}(30|21|03)$
$-\frac{4}{3}(21|21|12)$
$-\frac{1}{3}(30|12|12)$
$-\frac{37}{270}(20|12)$
$+\frac{2}{27}(30|02)$
$+\frac{2}{27}(21|11)$</td><td>$\frac{5}{6}(30|21|20|03|02)$
$+\frac{40}{9}(30|21|03)$
$+\frac{5}{3}(30|12|12)$
$+\frac{5}{2}(21|21|12)$
$+\frac{2}{9}(20|12)$
$-\frac{17}{18}(21|11)$
$-\frac{13}{36}(30|02)$</td></tr>
<tr><th>$-\frac{1}{3}((11|11)|30|21|03)$</th><th>$\frac{1}{6}((11|02)|30|21|12)$</th><th>$\frac{1}{6}((20|20)|12|12|12)$</th><th>$\frac{5}{6}((20|02)|30|12|12)$</th></tr>
<tr><td>$-\frac{1}{3}(30|21|11|11|03)$
$+\frac{5}{9}(30|21|03)$
$+\frac{2}{45}(20|12)$
$-\frac{7}{90}(30|02)$
$-\frac{7}{45}(21|11)$</td><td>$\frac{1}{6}(30|21|12|11|02)$
$-\frac{1}{3}(21|21|12)$
$-\frac{1}{9}(30|12|12)$
$+\frac{11}{540}(20|12)$
$-\frac{5}{216}(30|02)$
$-\frac{5}{108}(21|11)$</td><td>$\frac{1}{6}(20|20|12|12|12)$
$-\frac{2}{3}(30|12|12)$
$-\frac{4}{3}(21|21|12)$
$+\frac{37}{135}(20|12)$
$-\frac{4}{27}(30|02)$
$-\frac{4}{27}(21|11)$</td><td>$\frac{5}{6}(30|20|12|12|02)$
$+\frac{85}{18}(30|12|12)$
$+\frac{10}{3}(21|21|12)$
$+\frac{10}{9}(30|21|03)$
$-\frac{11}{9}(20|12)$
$+\frac{13}{27}(30|02)$
$+\frac{2}{9}(21|11)$</td></tr>
<tr><th>$-\frac{1}{3}((11|11)|30|12|12)$</th><th>$\frac{1}{6}((11|02)|21|21|21)$</th><th>$\frac{5}{6}((20|02)|21|21|12)$</th><th>$-\frac{1}{3}((11|11)|21|21|12)$</th></tr>
<tr><td>$-\frac{1}{3}(30|12|12|11|11)$
$+\frac{1}{3}(30|12|12)$</td><td>$\frac{1}{6}(21|21|21|11|02)$
$-\frac{1}{3}(21|21|12)$
$-\frac{1}{27}(20|12)$
$+\frac{1}{54}(30|02)$
$+\frac{1}{135}(21|11)$</td><td>$\frac{5}{6}(21|21|20|12|02)$
$+\frac{145}{18}(21|21|12)$
$+\frac{10}{9}(30|12|12)$
$+\frac{5}{9}(30|21|03)$
$+\frac{32}{81}(20|12)$
$+\frac{17}{81}(21|11)$
$-\frac{11}{162}(30|02)$</td><td>$-\frac{1}{3}(21|21|12|11|11)$
$+\frac{1}{9}(21|21|12)$
$-\frac{2}{81}(20|12)$
$+\frac{1}{81}(30|02)$
$+\frac{2}{405}(21|11)$</td></tr>
</table>

Table 29: Composed Weyl-ordered products from $[\mathcal{C}_\gamma, 21] = 0$.

<table>
<tr><th>$\frac{1}{6}((20|20)|12|02)$</th><th>$\frac{5}{6}((20|02)|30|02)$</th><th>$-\frac{1}{3}((11|11)|30|02)$</th><th>$\frac{1}{6}((20|20)|11|03)$</th><th>$\frac{1}{6}((11|02)|30|11)$</th></tr>
<tr><td>$\frac{1}{6}(20|20|12|02)$
$-\frac{2}{9}(30|02)$
$-\frac{2}{9}(20|12)$
$-\frac{4}{9}(21|11)$</td><td>$\frac{5}{6}(30|20|02|02)$
$+\frac{25}{18}(30|02)$
$+\frac{5}{3}(21|11)$</td><td>$-\frac{1}{3}(30|11|11|02)$
$+\frac{7}{18}(30|02)$</td><td>$\frac{1}{6}(20|20|11|03)$
$-\frac{2}{3}(21|11)$
$-\frac{1}{3}(20|12)$</td><td>$\frac{1}{6}(30|11|11|02)$
$-\frac{1}{3}(21|11)$
$-\frac{1}{36}(30|02)$</td></tr>
<tr><th>$\frac{1}{6}((11|02)|21|20)$</th><th>$\frac{5}{6}((20|02)|21|11)$</th><th>$-\frac{1}{3}((11|11)|21|11)$</th><th>$\frac{5}{6}((20|02)|20|12)$</th><th>$-\frac{1}{3}((11|11)|20|12)$</th></tr>
<tr><td>$\frac{1}{6}(21|20|11|02)$
$-\frac{1}{6}(21|11)$
$-\frac{1}{9}(20|12)$</td><td>$\frac{5}{6}(21|20|11|02)$
$+\frac{55}{18}(21|11)$
$+\frac{5}{18}(30|02)$
$+\frac{5}{9}(20|12)$</td><td>$-\frac{1}{3}(21|11|11|11)$
$+\frac{1}{18}(21|11)$</td><td>$\frac{5}{6}(20|20|12|02)$
$+\frac{5}{2}(20|12)$
$+\frac{10}{9}(21|11)$</td><td>$-\frac{1}{3}(20|12|11|11)$
$+\frac{1}{6}(20|12)$</td></tr>
</table>

Table 30: Composed Weyl-ordered products from $[\mathcal{C}_\epsilon, 21] = 0$.

A.8 Commutators and Interaction g

The main purpose of this section is to sketch in a detailed way the necessary steps when calculating commutators which include the interaction g. From the previous considerations, the main difference lies in the use of π_k operators possessing the fundamental property

$$[\pi_j, \pi_k] = 0 \ , \tag{A.183}$$

which simplifies many calculations, but since they are defined in terms of permutation operators (see Section A.2), one has to carefully deal with them.

Let us start with the usual operators B_{kl}, including COM; these operators will act on the space of all completely symmetric functions under permutation of two particles [1]. Firstly, we prove that

$$\tilde{B}_{02} = res\left(\sum_{a=1}^{3} \pi_a^2\right) = B_{02} + 2g(g-1)\sum_{k<l}\frac{1}{x_{kl}^2} \ . \tag{A.184}$$

Let ψ be any such function. Starting from

$$\begin{aligned} \pi_a\psi &= p_a\psi + ig\sum_{b(\neq a)}^{3}\frac{1}{x_{ab}}s_{ab}\psi \\ &= p_a\psi + ig\sum_{b(\neq a)}^{3}\frac{1}{x_{ab}}\psi \ , \end{aligned} \tag{A.185}$$

and applying again the operator π_a on this result, we get successively:

$$\pi_a^2\psi = p_a^2\psi + ig\sum_{b(\neq a)}^{3} p_a\left[\frac{1}{x_{ab}}\psi\right] + ig\sum_{c(\neq a)}^{3}\frac{1}{x_{ac}}s_{ac}\left[p_a\psi\right] + ig\sum_{c(\neq a)}^{3}\frac{1}{x_{ac}}s_{ac}\left[ig\sum_{b(\neq a)}^{3}\frac{1}{x_{ab}}\psi\right] \ . \tag{A.186}$$

Using the definition $p_\alpha = -i\partial_\alpha$ it is clear

$$p_\alpha\left[\frac{1}{x_{\beta\gamma}}\right] = \frac{i}{x_{\beta\gamma}^2}\left(\delta_{\beta\alpha} - \delta_{\gamma\alpha}\right) \ . \tag{A.187}$$

Applying this term by term in (A.186):

$$\sum_{b(\neq a)}^{3} p_a\left[\frac{1}{x_{ab}}\psi\right] = +i\sum_{b(\neq a)}^{3}\frac{1}{x_{ab}^2}\psi + \sum_{b(\neq a)}^{3}\frac{1}{x_{ab}}p_a\left(\psi\right) \ , \tag{A.188}$$

$$\sum_{c(\neq a)}^{3}\frac{1}{x_{ac}}s_{ac}\left[p_a\psi\right] = \sum_{c(\neq a)}^{3}\frac{1}{x_{ac}}p_c\left(\psi\right) \ , \tag{A.189}$$

$$\begin{aligned} \sum_{c(\neq a)}^{3}\frac{1}{x_{ac}}s_{ac}\left[\sum_{b(\neq a)}^{3}\frac{1}{x_{ab}}\psi\right] &= \sum_{c(\neq a)}^{3}\frac{1}{x_{ac}}s_{ac}\left[\frac{1}{x_{ac}}\psi\right] + \sum_{b\neq c(\neq a)}^{3}\frac{1}{x_{ac}}s_{ac}\left[\frac{1}{x_{ab}}\psi\right] \\ &= -\sum_{c(\neq a)}^{3}\frac{1}{x_{ac}^2}\psi + \sum_{b\neq c(\neq a)}^{3}\frac{1}{x_{ac}}\frac{1}{x_{cb}}\psi \ . \end{aligned} \tag{A.190}$$

Inserting (A.188), (A.189) and (A.190) in (A.186):

$$\pi_a^2\psi = p_a^2\psi - g\sum_{b(\neq a)}^{3}\frac{1}{x_{ab}^2}\psi + ig\sum_{b(\neq a)}^{3}\frac{p_a+p_b}{x_{ab}}\psi + g^2\sum_{c(\neq a)}^{3}\frac{1}{x_{ac}^2}\psi - g^2\sum_{b\neq c(\neq a)}^{3}\frac{1}{x_{ac}}\frac{1}{x_{cb}}\psi$$

$$= p_a^2\psi + g(g-1)\sum_{b(\neq a)}^{3}\frac{1}{x_{ab}^2}\psi + ig\sum_{b(\neq a)}^{3}\frac{p_a+p_b}{x_{ab}}\psi - g^2\sum_{b\neq c(\neq a)}^{3}\frac{1}{x_{ac}}\frac{1}{x_{cb}}\psi \ . \quad \text{(A.191)}$$

Inserting this last result in the definition (A.184), and realizing that

$$\sum_{a}^{3}\sum_{b(\neq a)}^{3}\frac{p_a+p_b}{x_{ab}}\psi = 0 \ , \quad \text{(A.192)}$$

$$\sum_{a}^{3}\sum_{b\neq c(\neq a)}^{3}\frac{1}{x_{ac}}\frac{1}{x_{cb}}\psi = 0 \ , \quad \text{(A.193)}$$

we finally obtain

$$\tilde{B}_{02}\psi = B_{02}\psi + +g(g-1)\sum_{a}\sum_{b\neq a}\frac{1}{x_{ab}^2}\psi = B_{02}\psi + 2g(g-1)\sum_{a<b}^{3}\frac{1}{x_{ab}^2}\psi \ , \quad \text{(A.194)}$$

$$\Longrightarrow \tilde{B}_{02} = B_{02} + 2g(g-1)\sum_{a<b}^{3}\frac{1}{x_{ab}^2} \ . \quad \text{(A.195)}$$

For the rest of the operators the procedure is the same, just being more complicated in the cases $\tilde{B}_{03}$ and $\tilde{B}_{12}$ because of the higher order derivatives.

As a second step, COM decoupling, the idea is practically the same: the old B_{kl} operators are promoted to $\tilde{B}_{kl}$, and the transformation is applied. The only difficulty lies in the cumbersome algebraic calculations associated when expanding Weyl-ordered products, etc. but in principle is the same as in the free case. We only show how this be done for the commutator $[\tilde{30},\tilde{03}]$. Using the definitions

$$\tilde{30} = 30 \ , \quad \text{(A.196)}$$

$$\tilde{03} = 03 + 3g(g-1)\sum_{k<l}\frac{p_k+p_l}{x_{kl}^2} \ , \quad \text{(A.197)}$$

we have:

$$[\tilde{30},\tilde{03}] = [30,03] + 3g(g-1)\sum_{k<l}\left[30,\frac{p_k+p_l}{x_{kl}^2}\right] \ . \quad \text{(A.198)}$$

Let us focus on the term

$$\left[30,\frac{p_k+p_l}{x_{kl}^2}\right] = \sum_{a=1}^{3}\left[x_a^3,\frac{p_k+p_l}{x_{kl}^2}\right] = \sum_{a=1}^{3}\left[x_a^3,p_k+p_l\right]\frac{1}{x_{kl}^2} = 3i\frac{x_k^2+x_l^2}{x_{kl}^2} \ . \quad \text{(A.199)}$$

This last result in eq. (A.198) yields:

$$[\tilde{30},\tilde{03}] = [30,03] + 9ig(g-1)\sum_{k<l}\frac{x_k^2+x_l^2}{x_{kl}^2} = 9i(22) + 9i + 9ig(g-1)\sum_{k<l}\frac{x_k^2+x_l^2}{x_{kl}^2} \ . \quad \text{(A.200)}$$

The trick now consists in expressing the operator B_{22} in terms of tilde operators. Recalling the definition of B_{22}

$$\begin{aligned} B_{22} = \frac{2}{3}(21|01) + \frac{1}{6}(20|02) + \frac{2}{3}(12|10) + \frac{1}{3}(11|11) + \\ - \frac{1}{6}(20|01|01) - \frac{2}{3}(11|10|01) - \frac{1}{6}(10|10|02) + \\ + \frac{1}{6}(10|10|01|01) - \frac{3}{2} , \end{aligned} \tag{A.201}$$

it is clear, that only the terms (20|02), (12|10) and (10|10|02) will be modified:

$$\frac{1}{6}(20|02) = \frac{1}{6}(\tilde{20}|\tilde{02}) - \frac{g(g-1)}{3} \sum_{a<b} \frac{\tilde{20}}{x_{ab}^2} , \tag{A.202}$$

$$\frac{2}{3}(12|10) = \frac{2}{3}(\tilde{12}|\tilde{10}) - \frac{2g(g-1)}{3} \sum_{a<b} \frac{(x_a + x_b)(\tilde{10})}{x_{ab}^2} , \tag{A.203}$$

$$-\frac{1}{6}(10|10|02) = -\frac{1}{6}(\tilde{10}|\tilde{10}|\tilde{02}) + \frac{g(g-1)}{3} \sum_{a<b} \frac{(\tilde{10})^2}{x_{ab}^2} . \tag{A.204}$$

With these expressions, the commutator (A.200) now takes the form:

$$\begin{aligned} i^{-1}\left[\tilde{30}, \tilde{03}\right] &= 6(\tilde{21}|\tilde{01}) + \frac{3}{2}(\tilde{20}|\tilde{02}) + 6(\tilde{12}|\tilde{10}) + 3(\tilde{11}|\tilde{11}) - \frac{3}{2}(\tilde{20}|\tilde{01}|\tilde{01}) - 6(\tilde{11}|\tilde{10}|\tilde{01}) \\ &\quad - \frac{3}{2}(\tilde{10}|\tilde{10}|\tilde{02}) + \frac{3}{2}(\tilde{10}|\tilde{10}|\tilde{01}|\tilde{01}) - \frac{27}{2} + \\ &\quad - 3g(g-1) \sum_{a<b} \frac{(\tilde{20})}{x_{ab}^2} - 6g(g-1) \sum_{a<b} \frac{(x_a + x_b)(\tilde{10})}{x_{ab}^2} + 3g(g-1) \sum_{a<b} \frac{(\tilde{10})^2}{x_{ab}^2} \\ &\quad + 9 + 9g(g-1) \sum_{a<b} \frac{x_a^2 + x_b^2}{x_{ab}^2} . \end{aligned} \tag{A.205}$$

Now, we need to prove that the extra terms appearing in the last expression either vanish or combine into a constant. This will be achieved again by direct calculation:

$$9ig(g-1) \sum_{a<b}^{3} \frac{x_a^2 + x_b^2}{x_{ab}^2} = 9ig(g-1) \left[\frac{x_1^2 + x_2^2}{x_{12}^2} + \frac{x_1^2 + x_3^2}{x_{13}^2} + \frac{x_2^2 + x_3^2}{x_{23}^2} \right] , \tag{A.206}$$

$$-3g(g-1) \sum_{a<b} \frac{(\tilde{20}) - (\tilde{10})^2}{x_{ab}^2} = -6g(g-1)\left(-x_1x_2 - x_1x_3 - x_2x_3\right) \left(\frac{1}{x_{12}^2} + \frac{1}{x_{13}^2} + \frac{1}{x_{23}^2} \right) , \tag{A.207}$$

$$\begin{aligned} & - 6g(g-1) \sum_{a<b} \frac{(x_a + x_b)(\tilde{10})}{x_{ab}^2} \\ &\quad = -6g(g-1) \left(\frac{x_1 + x_2}{x_{12}^2} + \frac{x_1 + x_3}{x_{13}^2} + \frac{x_2 + x_3}{x_{23}^2} \right) (x_1 + x_2 + x_3) \\ &\quad = -6g(g-1) \left(\frac{(x_1 + x_2)^2 + x_3(x_1 + x_2)}{x_{12}^2} + \right. \\ &\quad \left. + \frac{(x_1 + x_3)^2 + x_2(x_1 + x_3)}{x_{13}^2} + \frac{(x_2 + x_{23})^2 + x_1(x_2 + x_3)}{x_{23}^2} \right) . \end{aligned} \tag{A.208}$$

Take now only the terms proportional to x_{12}^{-2}:

$$
\begin{aligned}
-\frac{6g(g-1)}{x_{12}^2}\left[(x_1+x_2)^2+x_3(x_1+x_2)+(-x_1x_2-x_1x_3-x_2x_3)\right]+9g(g-1)\frac{x_1^2+x_2^2}{x_{12}^2} =\\
=\frac{g(g-1)}{x_{12}^2}\left[9(x_1^2+x_2^2)-6(x_1+x_2)^2-6x_3(x_1+x_2)-6(-x_1x_2-x_1x_3-x_2x_3)\right]\\
=\frac{g(g-1)}{x_{12}^2}\left[9x_1^2+9x_2^2-6x_1^2-12x_1x_2-6x_2^2+6x_1x_2\right]\\
=\frac{g(g-1)}{x_{12}^2}\left[3x_1^2+3x_2^2-6x_1x_2\right]\\
=\frac{g(g-1)}{x_{12}^2}\cdot 3\,(x_1-x_2)^2=3g(g-1)\ .
\end{aligned}
\tag{A.209}
$$

Repeating the procedure for the contributions proportional to x_{13}^{-2} and x_{23}^2, adding the respective results with (A.209) and putting in (A.205), the final expression is:

$$
\begin{aligned}
i^{-1}\left[\tilde{30},\tilde{03}\right]=6(\tilde{21}|\tilde{01})+\frac{3}{2}(\tilde{20}|\tilde{02})+6(\tilde{12}|\tilde{10})+3(\tilde{11}|\tilde{11})-\frac{3}{2}(\tilde{20}|\tilde{01}|\tilde{01})-6(\tilde{11}|\tilde{10}|\tilde{01})+\\
-\frac{3}{2}(\tilde{10}|\tilde{10}|\tilde{02})+\frac{3}{2}(\tilde{10}|\tilde{10}|\tilde{01}|\tilde{01})-\frac{9}{2}+9g(g-1)\ .
\end{aligned}
\tag{A.210}
$$

From this point onwards, the treatment is exactly the same as in the case without interaction: we decouple the center of mass and total momentum using the already known transformation now with $\tilde{B}_{kl}$ operators:

* Level 1 operators:

$$
\left.\begin{aligned}
(\tilde{10}') &= (\tilde{10})\\
(\tilde{01}') &= (\tilde{01})\\
(\tilde{00}') &= (\tilde{00})=N=3
\end{aligned}\right\}\ .
\tag{A.211}
$$

* Level 2 operators:

$$
\left.\begin{aligned}
(\tilde{20}') &= (\tilde{20})-\frac{1}{3}(\tilde{10}|\tilde{10})\\
(\tilde{11}') &= (\tilde{11})-\frac{1}{3}(\tilde{10}|\tilde{01})\\
(\tilde{02}') &= (\tilde{02})-\frac{1}{3}(\tilde{01}|\tilde{01})
\end{aligned}\right\}\ .
\tag{A.212}
$$

* Level 3 operators:

$$
\left.\begin{aligned}
(\tilde{30}') &= (\tilde{30})-(\tilde{20}|\tilde{10})+\frac{2}{9}(\tilde{10}|\tilde{10}|\tilde{10})\\
(\tilde{21}') &= (\tilde{21})-\frac{1}{3}(\tilde{20}|\tilde{01})-\frac{2}{3}(\tilde{11}|\tilde{10})+\frac{2}{9}(\tilde{10}|\tilde{10}|\tilde{01})\\
(\tilde{12}') &= (\tilde{12})-\frac{2}{3}(\tilde{11}|\tilde{01})-\frac{1}{3}(\tilde{10}|\tilde{02})+\frac{2}{9}(\tilde{10}|\tilde{01}|\tilde{01})\\
(\tilde{03}') &= (\tilde{03})-(\tilde{02}|\tilde{01})+\frac{2}{9}(\tilde{01}|\tilde{01}|\tilde{01})
\end{aligned}\right\}\ .
\tag{A.213}
$$

$$
\begin{aligned}
[\tilde{30}', \tilde{03}'] = [\tilde{30}, \tilde{03}] - [\tilde{30}, (\tilde{02}|\tilde{01})] + \frac{2}{9} \cdot [\tilde{30}, (\tilde{01}|\tilde{01}|\tilde{01})] \\
- [(\tilde{20}|\tilde{10}), \tilde{03}] + [(\tilde{20}|\tilde{10}), (\tilde{02}|\tilde{01})] - \frac{2}{9} \cdot [(\tilde{20}|\tilde{10}), (\tilde{01}|\tilde{01}|\tilde{01})] \\
+ \frac{2}{9} \cdot [(\tilde{10}|\tilde{10}|\tilde{10}), \tilde{03}] - \frac{2}{9} \cdot [(\tilde{10}|\tilde{10}|\tilde{10}), (\tilde{02}|\tilde{01})] + \frac{4}{81} \cdot [(\tilde{10}|\tilde{10}|\tilde{10}), (\tilde{01}|\tilde{01}|\tilde{01})] \ . \quad \text{(A.214)}
\end{aligned}
$$

The first commutator $[\tilde{30}, \tilde{03}]$ was calculated in (A.210); re-expressing this commutator in terms of the new operators $\tilde{B}'_{kl}$, we arrive to one of the central results of the present work:

$$
\left[\tilde{30}', \tilde{03}'\right] = -\frac{3}{2} \left(\tilde{20}'|\tilde{02}'\right) + 3(\tilde{11}'|\tilde{11}') - 4 + 9g(g-1) \ . \quad \text{(A.215)}
$$

The treatment for $\left[\tilde{21}', \tilde{12}'\right]$ and the rest of the commutators is very similar and straightforward, but lengthy due to algebraic difficulties.

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