Response function from thermal Greens

(Kritik : Quelle, Antwort und response function sind hier leider eichabhängig.)

Consider the QCD Lagrangian (including massless quarks, gauge fixing and ghosts) and add a source term $\mathcal{L}_{\rm ext} = J^{\nu b} A^b_\nu$. We study $<A^a_\mu(x)>_\bullet$ as one of the possibly nontrivial responses of the system (H). The index $\bullet$ refers to the full system $H_\bullet := H + H_{\rm ext}$, $H_{\rm ext} = -\int d^3r\, J^{\nu b} A^b_\nu$; $A(x)$ are Heisenberg operators with respect to H, but are in the interaction picture with respect to $H_\bullet$; $<...>_\bullet :=$ tr $(\sigma...)$, $<...> :=$ tr $(\rho_o...)$, $\rho_o := e^{\beta(F-H)}$, $F = -T\ln(Z)$.

$\dot\rho = -i\,[H_\bullet, \rho]$ (Schödinger picture), $\sigma = e^{iHt}\rho e^{-iHt} \;\Rightarrow\; \dot\sigma = -i\,[H_{\rm ext}, \sigma]$, $\sigma(-\infty) = \rho_o$:

$$<A^a_\mu(x)>_\bullet = \text{ tr } (A^a_\mu(x)\,\sigma) = -i\int_{-\infty}^t dt' \text{ tr } (A^a_\mu(x)[H_{\rm ext}(t'), \rho_o]) + O(J^2)$$

$$= i\int_{-\infty}^t dt' \int d^3r' <[A^a_\mu(x), A^b_\nu(x')]> J^{\nu b}(x') + O(J^2)$$

Let $J(x) = e^{\epsilon t} j(x)$. The system answers by $<A(x)>= e^{\epsilon t} a(x)$. $j(x)$ and $a(x)$ are the quantities "in the presence", the relation between we are really interested in. Thus we define the response function χ by $a(x) = \int d^4x' \chi(x,x') j(x')$ and obtain

$$\chi^{ab}_{\mu\nu}(x,x') = i\theta(t-t')e^{-\epsilon(t-t')} < [A^a_\mu(x), A^b_\nu(x')] >=: \chi^{ab}_{\nu\mu}(x-x')\ ,$$

where the last relation is due to translational invariance. Fourier transforming by $(2\pi)^{-4}\int d^4K e^{-iKx}...$ we have $\widetilde a(K) = \widetilde\chi(K)\,\widetilde j(K)$ $\;(K := (\omega, \vec k\,)$, metrics $+---$) and

$$\widetilde\chi^{ab}_{\mu\nu}(K) \;=\; \int dt \int d^3r e^{i\omega t - i\vec k\vec r} i\theta(t) e^{-\epsilon t} < [A^a_\mu(x), A^b_\nu(0)] >$$

$$A^a_\mu(x) = e^{iHt}(2\pi)^{-3}\int d^3q e^{i\vec q\vec r}\widetilde A^a_\mu(\vec q\,)e^{-iHt}$$

$$= i\int_0^\infty dt e^{(i\omega-\epsilon)t}(2\pi)^{-3}\int d^3p \text{ tr } [e^{\beta(F-H)}(e^{iHt}\widetilde A^a_\mu(\vec k\,)e^{-iHt}\widetilde A^b_\nu(\vec p) - \widetilde A^b_\nu(\vec p)e^{iHt}\widetilde A^a_\mu(\vec k\,)e^{-iHt})]$$

$$\omega_{fg} := E_f - E_g \ :$$

$$= i\int_0^\infty dt e^{(i\omega-\epsilon)t}(2\pi)^{-3}\int d^3p \sum_{f,g} e^{-i\omega_{fg}t}\,[e^{\beta(F-E_g)} - e^{\beta(F-E_f)}] <f|\widetilde A^b_\nu(\vec p)|g><g|\widetilde A^a_\mu(\vec k\,)|f>$$

$$= \int dx\, \frac{1}{x-\omega-i\epsilon}\ \mathcal{A}^{ab}_{\mu\nu}(x, \vec k\,) \qquad \text{with}$$

$$\mathcal{A}^{ab}_{\mu\nu}(x, \vec k\,) \;=\; (2\pi)^{-3}\int d^3p \sum_{f,g} e^{\beta(F-E_f)}\,[e^{\beta\omega_{fg}} - 1]\ \delta(x-\omega_{fg}) <f|\widetilde A^b_\nu(\vec p)|g><g|\widetilde A^a_\mu(\vec k\,)|f>$$

We might compare this spectral representation of the response function with that of the Matsubara thermal Greensfunction $G^{ab}_{\mu\nu}(x) =<\mathcal{T} A^a_\mu(x) A^b_\nu(0)>$ where here $x := (-i\tau, \vec r\,)$ and (in the following) $K := (i\omega_n, \vec k\,)$, $\omega_n := 2\pi nT$. $\mathcal{T}$ orders larger τ to the left.

$$\widetilde G^{ab}_{\mu\nu}(K) \;=\; \int d^3r \int_0^\beta d\tau e^{iKx} G^{ab}_{\mu\nu}(x)$$

$$= \int_0^\beta d\tau e^{i\omega_n\tau}\int d^3r e^{-i\vec k\vec r} \text{ tr } [e^{\beta(F-H)}e^{H\tau}(2\pi)^{-3}\int d^3q e^{i\vec q\vec r}\widetilde A^a_\mu(\vec q\,)e^{-H\tau}(2\pi)^{-3}\int d^3p \widetilde A^b_\nu(\vec p\,)]$$

$$= \sum_{fg}\frac{1}{\omega_{fg}-i\omega_n}e^{\beta(F-E_f)}\,[e^{\beta\omega_{fg}} - 1]\ (2\pi)^{-3}\int d^3p <f|\widetilde A^b_\nu(\vec p)|g><g|\widetilde A^a_\mu(\vec k\,)|f>$$

$$= \int dx\, \frac{1}{x - i\omega_n}\ \mathcal{A}^{ab}_{\mu\nu}(x, \vec k\,)$$

The relation between the response function and the thermal Green is now obvious:

$$\widetilde\chi^{ab}_{\mu\nu}(\omega, \vec k\,) \;=\; \widetilde G^{ab}_{\mu\nu}(\; i\omega_n \longrightarrow \omega + i\epsilon\ ,\ \vec k\,)$$

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